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# Real Number System

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**ABSTRACT: Real numbers** are simply the combination of rational and irrational numbers, in the number system. In general, all the arithmetic operations can be performed on these numbers and they can be represented in the number line, also. At the same time, the **imaginary numbers** are the un-real numbers, which cannot be expressed in the number line and are commonly used to represent a **complex number**. Some of the examples of real numbers are 23, -12, 6.99, 5/2, π, and so on. In this article, we are going to discuss the definition of real numbers, the properties of real numbers and the examples of real numbers with complete explanations.

**KEYWORDS:** real, number, imaginary, complex, rational, irrational, arithmetic, operations, system

## I.INTRODUCTION

Real numbers can be defined as the union of both rational and irrational numbers. They can be both positive or negative and are denoted by the symbol "R". All the natural numbers, decimals and fractions come under this category. See the figure, given below, which shows the classification of real numerals.

The set of real numbers consists of different categories, such as natural and whole numbers, integers, rational and irrational numbers. In the table given below, all the real numbers formulas (i.e.) the representation of the classification of real numbers are defined with examples.

| **Category** | **Definition** | **Example** |
|---|---|---|
| Natural Numbers | Contain all counting numbers which start from 1. N = {1, 2, 3, 4,……} | All numbers such as 1, 2, 3, 4, 5, 6,…..… |
| Whole Numbers | Collection of zero and natural numbers. W = {0, 1, 2, 3,…..} | All numbers including 0 such as 0, 1, 2, 3, 4, 5, 6,…..… |
| Integers | The collective result of whole numbers and negative of all natural numbers. | Includes: -infinity (-∞),……..-4, -3, -2, -1, 0, 1, 2, 3, 4, ……+infinity (+∞) |
| Rational Numbers | Numbers that can be written in the form of p/q, where q≠0. | Examples of rational numbers are ½, 5/4 and 12/6 etc. |
| Irrational Numbers | The numbers which are not rational and cannot be written in the form of p/q. | Irrational numbers are non-terminating and non-repeating in nature like √2. |

Properties of Real Numbers

The following are the four main properties of real numbers:

- Commutative property

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- Associative property
- Distributive property
- Identity property

Consider "m, n and r" are three real numbers. Then the above properties can be described using m, n, and r as shown below:

**Commutative Property**

If m and n are the numbers, then the general form will be m + n = n + m for addition and m.n = n.m for multiplication.

- **Addition:** m + n = n + m. For example, 5 + 3 = 3 + 5, 2 + 4 = 4 + 2.
- **Multiplication:** m × n = n × m. For example, 5 × 3 = 3 × 5, 2 × 4 = 4 × 2.

**Associative Property**

If m, n and r are the numbers. The general form will be m + (n + r) = (m + n) + r for addition(mn) r = m (nr) for multiplication.

- **Addition:** The general form will be m + (n + r) = (m + n) + r. An example of additive associative property is 10 + (3 + 2) = (10 + 3) + 2.
- **Multiplication:** (mn) r = m (nr). An example of a multiplicative associative property is (2 × 3) 4 = 2 (3 × 4).

**Distributive Property**

For three numbers m, n, and r, which are real in nature, the distributive property is represented as:

m (n + r) = mn + mr and (m + n) r = mr + nr.

- Example of distributive property is: 5(2 + 3) = 5 × 2 + 5 × 3. Here, both sides will yield 25.

**Identity Property**

There are additive and multiplicative identities.

- **For addition:** m + 0 = m. (0 is the additive identity)
- **For multiplication:** m × 1 = 1 × m = m. (1 is the multiplicative identity)

The common concepts introduced include representing real numbers on a number line, operations on real numbers, properties of real numbers, and the law of exponents for real numbers. In Class 10, some advanced concepts related to real numbers are included. Apart from what are real numbers, students will also learn about the real numbers formulas and concepts such as Euclid's Division Lemma, Euclid's Division Algorithm and the fundamental theorem of arithmetics

## II.DISCUSSION

In mathematics, a **real number** is a number that can be used to measure a continuous one-dimensional quantity such as a distance, duration or temperature. Here, continuous means that values can have arbitrarily small variations. Every real number can be almost uniquely represented by an infinite decimal expansion.

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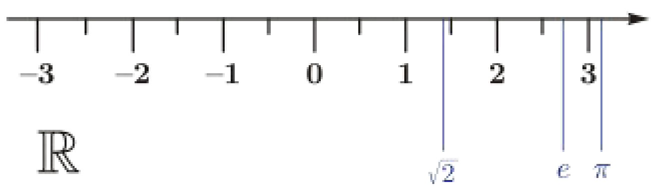
The real numbers are fundamental in calculus (and more generally in all mathematics), in particular by their role in the classical definitions of limits, continuity and derivatives.

The set of real numbers is denoted **R** and is sometimes called "the reals".The adjective real in this context was introduced in the 17th century by René Descartes to distinguish real numbers, associated with physical reality, from imaginary numbers (such as the square roots of −1), which seemed like a theoretical contrivance unrelated to physical reality.

The real numbers include the rational numbers, such as the integer −5 and the fraction 4/3. The rest of the real numbers are called irrational numbers, and include algebraic numbers (such as the square root $\sqrt{2}$ = 1.414...) and transcendental numbers (such as $\pi$ = 3.1415...).

Real numbers can be thought of as all points on an infinitely long line called the number line or real line, where the points corresponding to integers (..., −2, −1, 0, 1, 2, ...) are equally spaced.




Conversely, analytic geometry is the association of points on lines (especially axis lines) to real numbers such that geometric displacements are proportional to differences between corresponding numbers.

The informal descriptions above of the real numbers are not sufficient for ensuring the correctness of proofs of theorems involving real numbers. The realization that a better definition was needed, and the elaboration of such a definition was a major development of 19th-century mathematics and is the foundation of real analysis, the study of real functions and real-valued sequences. A current axiomatic definition is that real numbers form the unique (up to an isomorphism) Dedekind-complete ordered field. Other common definitions of real numbers include equivalence classes of Cauchy sequences (of rational numbers), Dedekind cuts, and infinite decimal representations. All these definitions satisfy the axiomatic definition and are thus equivalent.

**Basic properties**

- The real numbers include zero (0), the additive identity: adding 0 to any real number leaves that number unchanged: x + 0 = 0 + x = x.
- Every real number x has an additive inverse −x satisfying x + (−x) = −x + x = 0.
- The real numbers include a unit (1), the multiplicative identity: multiplying 1 by any real number leaves that number unchanged: 1 x = x 1 = x.
- Every nonzero real number x has a multiplicative inverse 1/x satisfying x(1/x) = (1/x)x = 1.
- Given any two real numbers x and y, the results of addition (x + y), subtraction (x − y), and multiplication (x y) are also real numbers, as is the result of division (x / y) if y is not zero. Thus the real numbers are closed under elementary arithmetic operations.
- The real numbers form a field.
- The real numbers are linearly ordered. For any distinct real numbers x and y, either x < y or y < x. If x < y and y < z then x < z. (See also inequality (mathematics).)
- Any nonzero real number x is either negative (x < 0) or positive (0 < x).
- The real numbers are an ordered field because the order < is compatible with addition and multiplication: if x < y then x + z < y + z; if 0 < x and 0 < y then 0 < x y. Because the square of any real number is non-negative, and the sum and product of non-negative real numbers is itself non-negative, non-negative real numbers are a positive cone of **R**.

- The real numbers make up an infinite set of numbers that cannot be injectively mapped to the infinite set of natural numbers, i.e., there are uncountably infinitely many real numbers, whereas the natural numbers are called countably infinite. This establishes that in some sense, there are more real numbers than there are elements in any countable set.
- Any nonempty bounded open interval (the set of all real numbers between two specified endpoints) can be mapped bijectively by an affine function (scaling and translation of the number line) to any other such interval. Every nonempty open interval contains uncountably infinitely many real numbers.
- The real numbers are unbounded. There is no greatest or least real number; the real numbers extend infinitely in both positive and negative directions.
- There is a hierarchy of countably infinite subsets of the real numbers, e.g., the integers, the rational numbers, the algebraic numbers and the computable numbers, each set being a proper subset of the next in the sequence. The complements of each of these sets in the reals (irrational, transcendental, and non-computable real numbers) is uncountably infinite.
- Real numbers can be used to express measurements of continuous quantities. They may be expressed by decimal representations, most of them having an infinite sequence of digits to the right of the decimal point; these are often represented like 324.823122147..., where the ellipsis indicates that infinitely many digits have been omitted.

More formally, the real numbers have the two basic properties of being an ordered field, and having the least upper bound property. The first says that real numbers comprise a field, with addition and multiplication as well as division by nonzero numbers, which can be totally ordered on a number line in a way compatible with addition and multiplication. The second says that, if a nonempty set of real numbers has an upper bound, then it has a real least upper bound. The second condition distinguishes the real numbers from the rational numbers: for example, the set of rational numbers whose square is less than 2 is a set with an upper bound (e.g. 1.5) but no (rational) least upper bound: hence the rational numbers do not satisfy the least upper bound property.

## III.RESULTS

A main reason for using real numbers is so that many sequences have limits. More formally, the reals are complete (in the sense of metric spaces or uniform spaces, which is a different sense than the Dedekind completeness of the order in the previous section):

A sequence $(x_n)$ of real numbers is called a Cauchy sequence if for any $\varepsilon > 0$ there exists an integer N (possibly depending on $\varepsilon$) such that the distance $|x_n - x_m|$ is less than $\varepsilon$ for all n and m that are both greater than N. This definition, originally provided by Cauchy, formalizes the fact that the $x_n$ eventually come and remain arbitrarily close to each other.

A sequence $(x_n)$ converges to the limit x if its elements eventually come and remain arbitrarily close to x, that is, if for any $\varepsilon > 0$ there exists an integer N (possibly depending on $\varepsilon$) such that the distance $|x_n - x|$ is less than $\varepsilon$ for n greater than N.

Every convergent sequence is a Cauchy sequence, and the converse is true for real numbers, and this means that the topological space of the real numbers is complete.

The set of rational numbers is not complete. For example, the sequence (1; 1.4; 1.41; 1.414; 1.4142; 1.41421; ...), where each term adds a digit of the decimal expansion of the positive square root of 2, is Cauchy but it does not converge to a rational number (in the real numbers, in contrast, it converges to the positive square root of 2).

The completeness property of the reals is the basis on which calculus, and, more generally mathematical analysis are built. In particular, the test that a sequence is a Cauchy sequence allows proving that a sequence has a limit, without computing it, and even without knowing it.

The real numbers are often described as "the complete ordered field", a phrase that can be interpreted in several ways.

First, an order can be lattice-complete. It is easy to see that no ordered field can be lattice-complete, because it can have no largest element (given any element z, z + 1 is larger).

Additionally, an order can be Dedekind-complete, see § Axiomatic approach. The uniqueness result at the end of that section justifies using the word "the" in the phrase "complete ordered field" when this is the sense of "complete" that is

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meant. This sense of completeness is most closely related to the construction of the reals from Dedekind cuts, since that construction starts from an ordered field (the rationals) and then forms the Dedekind-completion of it in a standard way.

These two notions of completeness ignore the field structure. However, an ordered group (in this case, the additive group of the field) defines a uniform structure, and uniform structures have a notion of completeness; the description in § Completeness is a special case. (We refer to the notion of completeness in uniform spaces rather than the related and better known notion for metric spaces, since the definition of metric space relies on already having a characterization of the real numbers). Simple fractions were used by the Egyptians around 1000 BC; the Vedic "Shulba Sutras" ("The rules of chords") in c. 600 BC include what may be the first "use" of irrational numbers. The concept of irrationality was implicitly accepted by early Indian mathematicians such as Manava (c. 750–690 BC), who was aware that the square roots of certain numbers, such as 2 and 61, could not be exactly determined.[7] Around 500 BC, the Greek mathematicians led by Pythagoras also realized that the square root of 2 is irrational.

The Middle Ages brought about the acceptance of zero, negative numbers, integers, and fractional numbers, first by Indian and Chinese mathematicians, and then by Arabic mathematicians, who were also the first to treat irrational numbers as algebraic objects (the latter being made possible by the development of algebra).[8] Arabic mathematicians merged the concepts of "number" and "magnitude" into a more general idea of real numbers.[9] The Egyptian mathematician Abū Kāmil Shujā ibn Aslam (c. 850–930) was the first to accept irrational numbers as solutions to quadratic equations, or as coefficients in an equation (often in the form of square roots, cube roots and fourth roots).[10] In Europe, such numbers, not commensurable with the numerical unit, were called irrational or surd ("deaf").

In the 16th century, Simon Stevin created the basis for modern decimal notation, and insisted that there is no difference between rational and irrational numbers in this regard.

In the 17th century, Descartes introduced the term "real" to describe roots of a polynomial, distinguishing them from "imaginary" ones.

In the 18th and 19th centuries, there was much work on irrational and transcendental numbers. Lambert (1761) gave a flawed proof that $\pi$ cannot be rational; Legendre (1794) completed the proof[11] and showed that $\pi$ is not the square root of a rational number.[12] Liouville (1840) showed that neither $e$ nor $e^2$ can be a root of an integer quadratic equation, and then established the existence of transcendental numbers; Cantor (1873) extended and greatly simplified this proof.[13] Hermite (1873) proved that $e$ is transcendental, and Lindemann (1882), showed that $\pi$ is transcendental. Lindemann's proof was much simplified by Weierstrass (1885), Hilbert (1893), Hurwitz,[14] and Gordan.[15]

The developers of calculus used real numbers without having defined them rigorously. The first rigorous definition was published by Cantor in 1871. In 1874, he showed that the set of all real numbers is uncountably infinite, but the set of all algebraic numbers is countably infinite. Cantor's first uncountability proof was different from his famous diagonal argument published in 1891.

## IV.CONCLUSIONS

**Applications in various fields:-**

**Physics**

In the physical sciences, most physical constants such as the universal gravitational constant, and physical variables, such as position, mass, speed, and electric charge, are modeled using real numbers. In fact, the fundamental physical theories such as classical mechanics, electromagnetism, quantum mechanics, general relativity and the standard model are described using mathematical structures, typically smooth manifolds or Hilbert spaces, that are based on the real numbers, although actual measurements of physical quantities are of finite accuracy and precision.Physicists have occasionally suggested that a more fundamental theory would replace the real numbers with quantities that do not form a continuum, but such proposals remain speculative.[17]

**Logic**

The real numbers are most often formalized using the Zermelo–Fraenkel axiomatization of set theory, but some mathematicians study the real numbers with other logical foundations of mathematics. In particular, the real numbers are

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also studied in reverse mathematics and in constructive mathematics.[18]The hyperreal numbers as developed by Edwin Hewitt, Abraham Robinson and others extend the set of the real numbers by introducing infinitesimal and infinite numbers, allowing for building infinitesimal calculus in a way closer to the original intuitions of Leibniz, Euler, Cauchy and others.Edward Nelson's internal set theory enriches the Zermelo–Fraenkel set theory syntactically by introducing a unary predicate "standard". In this approach, infinitesimals are (non-"standard") elements of the set of the real numbers (rather than being elements of an extension thereof, as in Robinson's theory).

**Computation**

Electronic calculators and computers cannot operate on arbitrary real numbers, because finite computers cannot directly store infinitely many digits or other infinite representations. Nor do they usually even operate on arbitrary definable real numbers, which are inconvenient to manipulate.Instead, computers typically work with finite-precision approximations called floating-point numbers, a representation similar to scientific notation. The achievable precision is limited by the data storage space allocated for each number, whether as fixed-point, floating-point, or arbitrary-precision numbers, or some other representation. Most scientific computation uses binary floating-point arithmetic, often a 64-bit representation with around 16 decimal digits of precision. Real numbers satisfy the usual rules of arithmetic, but floating-point numbers do not. The field of numerical analysis studies the stability and accuracy of numerical algorithms implemented with approximate arithmetic.Alternately, computer algebra systems can operate on irrational quantities exactly by manipulating symbolic formulas for rather than their rational or decimal approximation.[19] But exact and symbolic arithmetic also have limitations: for instance, they are computationally more expensive; it is not in general possible to determine whether two symbolic expressions are equal (the constant problem); and arithmetic operations can cause exponential explosion in the size of representation of a single number (for instance, squaring a rational number roughly doubles the number of digits in its numerator and denominator, and squaring a polynomial roughly doubles its number of terms), overwhelming finite computer storage.[20]A real number is called computable if there exists an algorithm that yields its digits. Because there are only countably many algorithms,[21] but an uncountable number of reals, almost all real numbers fail to be computable. Moreover, the equality of two computable numbers is an undecidable problem. Some constructivists accept the existence of only those reals that are computable. The set of definable numbers is broader, but still only countable.

**Set theory**

In set theory, specifically descriptive set theory, the Baire space is used as a surrogate for the real numbers since the latter have some topological properties (connectedness) that are a technical inconvenience. Elements of Baire space are referred to as "reals".underlying field is the field of the real numbers (or the real field). For example, real matrix, real polynomial and real Lie algebra. The word is also used as a noun, meaning a real number (as in "the set of all reals").

The real numbers can be generalized and extended in several different directions:

- The complex numbers contain solutions to all polynomial equations and hence are an algebraically closed field unlike the real numbers. However, the complex numbers are not an ordered field.[21]
- The affinely extended real number system adds two elements $+\infty$ and $-\infty$. It is a compact space. It is no longer a field, or even an additive group, but it still has a total order; moreover, it is a complete lattice.
- The real projective line adds only one value $\infty$. It is also a compact space. Again, it is no longer a field, or even an additive group. However, it allows division of a nonzero element by zero. It has cyclic order described by a separation relation.
- The long real line pastes together $\aleph_1^* + \aleph_1$ copies of the real line plus a single point (here $\aleph_1^*$ denotes the reversed ordering of $\aleph_1$) to create an ordered set that is "locally" identical to the real numbers, but somehow longer; for instance, there is an order-preserving embedding of $\aleph_1$ in the long real line but not in the real numbers. The long real line is the largest ordered set that is complete and locally Archimedean. As with the previous two examples, this set is no longer a field or additive group.
- Ordered fields extending the reals are the hyperreal numbers and the surreal numbers; both of them contain infinitesimal and infinitely large numbers and are therefore non-Archimedean ordered fields.[22]
- Self-adjoint operators on a Hilbert space (for example, self-adjoint square complex matrices) generalize the reals in many respects: they can be ordered (though not totally ordered), they are complete, all their eigenvalues are real and

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they form a real associative algebra. Positive-definite operators correspond to the positive reals and normal operators correspond to the complex numbers.[23]

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