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A Note on Two Summability Method

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ABSTRACT: In mathematics, a divergent series is an infinite series that is not convergent, meaning that the infinite sequence of the partial sums of the series does not have a finite limit.[1]

If a series converges, the individual terms of the series must approach zero. Thus any series in which the individual terms do not approach zero diverges. However, convergence is a stronger condition: not all series whose terms approach zero converge. A counterexample is the harmonic series. The divergence of the harmonic series was proven by the medieval mathematician Nicole Oresme.[2,3]

In specialized mathematical contexts, values can be objectively assigned to certain series whose sequences of partial sums diverge, in order to make meaning of the divergence of the series. A summability method or summation method is a partial function from the set of series to values. For example, Cesàro summation assigns Grandi's divergent series[4,5]

KEYWORDS: divergent, summability, method, two, harmonic, sequence, partial function, harmonic series, converge

I.INTRODUCTION

Before the 19th century, divergent series were widely used by Leonhard Euler and others, but often led to confusing and contradictory results. A major problem was Euler's idea that any divergent series should have a natural sum, without first defining what is meant by the sum of a divergent series. [6,7]Augustin-Louis Cauchy eventually gave a rigorous definition of the sum of a (convergent) series, and for some time after this, divergent series were mostly excluded from mathematics. They reappeared in 1886 with Henri Poincaré's work on asymptotic series. In 1890, Ernesto Cesàro realized that one could give a rigorous definition of the sum of some divergent series, and defined Cesàro summation[8,9]. (This was not the first use of Cesàro summation, which was used implicitly by Ferdinand Georg Frobenius in 1880; Cesàro's key contribution was not the discovery of this method, but his idea that one should give an explicit definition of the sum of a divergent series.) In the years after Cesàro's paper, several other mathematicians gave other definitions of the sum of a divergent series, although these are not always compatible: different definitions can give different answers for the sum of the same divergent series; so, when talking about the sum of a divergent series, it is necessary to specify which summation method one is using.[10,11,12]

A summability method M is regular if it agrees with the actual limit on all convergent series. Such a result is called an Abelian theorem for M, from the prototypical Abel's theorem. More subtle, are partial converse results, called Tauberian theorems, from a prototype proved by Alfred Tauber. Here partial converse means that if M sums the series Σ, and some side-condition holds, then Σ was convergent in the first place; without any side-condition such a result would say that M only summed convergent series (making it useless as a summation method for divergent series).[13,14,15]

The function giving the sum of a convergent series is linear, and it follows from the Hahn–Banach theorem that it may be extended to a summation method summing any series with bounded partial sums. This is called the Banach limit. This fact is not very useful in practice, since there are many such extensions, inconsistent with each other, and also since proving such operators exist requires invoking the axiom of choice or its equivalents, such as Zorn's lemma. They are therefore nonconstructive.

The subject of divergent series, as a domain of mathematical analysis, is primarily concerned with explicit and natural techniques such as Abel summation, Cesàro summation and Borel summation, and their relationships. The advent of Wiener's tauberian theorem marked an epoch in the subject, introducing unexpected connections to Banach algebra methods in Fourier analysis.[16,17]

Summation of divergent series is also related to extrapolation methods and sequence transformations as numerical techniques. Examples of such techniques are Padé approximants, Levin-type sequence transformations, and order-dependent mappings related to renormalization techniques for large-order perturbation theory in quantum mechanics.[18,19]

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II.DISCUSSION

Summation methods usually concentrate on the sequence of partial sums of the series. While this sequence does not converge, we may often find that when we take an average of larger and larger numbers of initial terms of the sequence, the average converges, and we can use this average instead of a limit to evaluate the sum of the series.[20,21] A summation method can be seen as a function from a set of sequences of partial sums to values. If A is any summation method assigning values to a set of sequences, we may mechanically translate this to a series-summation method A^{Σ} that assigns the same values to the corresponding series. There are certain properties it is desirable for these methods to possess if they are to arrive at values corresponding to limits and sums, respectively.[22,23]

- Regularity. A summation method is regular if, whenever the sequence s converges to x, A(s) = x. Equivalently, the corresponding series-summation method evaluates $A^{\Sigma}(a) = x$.
- Linearity. A is linear if it is a linear functional on the sequences where it is defined, so that A(k r + s) = k A(r) + A(s) for sequences r, s and a real or complex scalar k. Since the terms $a_{n+1} = s_{n+1} - s_n$ of the series a are linear functionals on the sequence s and vice versa, this is equivalent to A^{Σ} being a linear functional on the terms of the series.
- Stability (also called translativity). If s is a sequence starting from s_0 and s′ is the sequence obtained by omitting the first value and subtracting it from the rest, so that $s'_n = s_{n+1} - s_0$, then A(s) is defined if and only if A(s′) is defined, and $A(s) = s_0 + A(s')$. Equivalently, whenever $a'_n = a_{n+1}$ for all n, then $A^{\Sigma}(a) = a_0 + A^{\Sigma}(a')$.[1][2] Another way of stating this is that the shift rule must be valid for the series that are summable by this method.

The third condition is less important, and some significant methods, such as Borel summation, do not possess it.[3]

One can also give a weaker alternative to the last condition.

- Finite re-indexability. If a and a′ are two series such that there exists a bijection such that $a_i = a'_{f(i)}$ for all i, and if there exists some such that $a_i = a'_i$ for all i > N, then $A^{\Sigma}(a) = A^{\Sigma}(a')$. (In other words, a′ is the same series as a, with only finitely many terms re-indexed.) This is a weaker condition than stability, because any summation method that exhibits stability also exhibits finite re-indexability, but the converse is not true.)

A desirable property for two distinct summation methods A and B to share is consistency: A and B are consistent if for every sequence s to which both assign a value, A(s) = B(s). (Using this language, a summation method A is regular iff it is consistent with the standard sum Σ.) If two methods are consistent, and one sums more series than the other, the one summing more series is stronger.

There are powerful numerical summation methods that are neither regular nor linear, for instance nonlinear sequence transformations like Levin-type sequence transformations and Padé approximants, as well as the order-dependent mappings of perturbative series based on renormalization techniques.[24,25]

Taking regularity, linearity and stability as axioms, it is possible to sum many divergent series by elementary algebraic manipulations. This partly explains why many different summation methods give the same answer for certain series.[26,27]

III.RESULTS

The two classical summation methods for series, ordinary convergence and absolute convergence, define the sum as a limit of certain partial sums. These are included only for completeness; strictly speaking they are not true summation methods for divergent series since, by definition, a series is divergent only if these methods do not work. Most but not all summation methods for divergent series extend these methods to a larger class of sequences.

Absolute convergence

Absolute convergence defines the sum of a sequence (or set) of numbers to be the limit of the net of all partial sums $a_{k1} + ... + a_{kn}$, if it exists. It does not depend on the order of the elements of the sequence, and a classical theorem says that a sequence is absolutely convergent if and only if the sequence of absolute values is convergent in the standard sense.[28,29]

Sum of a series

Cauchy's classical definition of the sum of a series $a_0 + a_1 + ...$ defines the sum to be the limit of the sequence of partial sums $a_0 + ... + a_n$. This is the default definition of convergence of a sequence. (for positive values of the a_n) converges for large real s and can be analytically continued along the real line to s = −1, then its value at s = −1 is called the zeta regularized sum of the series $a_1 + a_2 + ...$ Zeta function regularization is nonlinear. In applications, the numbers a_i are sometimes the eigenvalues of a self-adjoint operator A with compact resolvent, and f(s) is then the trace of A^{-s}. For example, if A has eigenvalues 1, 2, 3, ... then f(s) is the Riemann zeta function, ζ(s), whose value at s = −1 is $-1/\boxed{12}$, assigning a value to the divergent series 1 + 2 + 3 + 4 + ... Other values of s can also be used to assign values for the divergent sums ζ(0) = 1 + 1 + 1 + ... = $-1/\boxed{2}$, ζ(−2) = 1 + 4 + 9 + ... = 0 and in general[31]

IV.CONCLUSIONS

The summability methods such as matrix summability, Cesàro summability, Hölder sum- mability, Harmonic summability, Generalized Cesàro summability, Riesz's typical means summability, Nörlund summability, Riesz's summability, generalized Nör- lund summability, indexed summability, Abel summability, Euler summability etc. The most well known proof of Weierstrass approximation theorem (see [1]) was given in [2,3]. Bernstein opened a new way by constructing a sequence of polynomials depending explicitly on evaluation of a function at rational values. Researchers have successfully extended this idea for approximating functions, for instance, L.V. Kantorovich introduced a new process to approximate Lebesgue integrable real-valued functions defined on [0,1] (see [4]). Recently, there has been an increasing degree of attention on approximation properties of Bernstein type operators with shape parameters (see [5,6,7,8,9,10,11,12]).

The decision on whether a sequence of positive linear operators converges strongly includes the use of Korovkin-type theorems. Using certain types of statistical convergences instead of usual convergence in Korovkin type approximation theory provides several benefits. The statistical convergence extends the scope of classical convergence of sequences of numbers or functions, and it has been used in various fields of mathematics such as summability theory [13], topology [14], optimization [15], measure theory [16], number theory [17], trigonometric series [18], approximation by positive linear operators [9,19,20,21,22,23,24,25]. Statistical convergence of double and single sequences were given in [26,27,28]. Unlike any convergent sequence, statistically convergent double or single sequences do not need to be bounded. This is why it is preferred to be used by many researchers in approximation theory (see, for instance, [29,30,31]).

The primary objective of this work is to establish a link between approximation theory and summability methods via four-dimensional matrices and construction of bivariate Bernstein-Kantorovich type operators on extended domain with reparametrized knots, as well as to prove some Korovkin theorems using two summability methods motivated by the studies [30,31] The first summability method is a statistical convergence concept which is stronger than the classical case and the second one is power series method (PSM). Since we create a link between the approximation theory and the summability theory we obtain the rate of convergence for PSM and the rate of statistical convergence by modulus of continuity (MC). Moreover, we provide some computer graphics to numerically analyze the efficiency and the accuracy of convergence of our operators, and obtain corresponding error and density plots. Finally, we provide some concluding remarks to emphasize main concepts of this article. All the results that have been obtained in the present paper can be extended for n-variate functions.

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