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Coupled Common Fixed Point Theorem for Two Maps in Modular Spaces

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ABSTRACT*:* In this paper, we obtain a coupled common fixed point theorem in modular spaces. We also give an example to illustrate our main theorem.

KEYWORDS*:* Modular spaces, Coupled fixed points, ρ - compatible mappings.

I. Introduction

The concept of a modular space was intiated by Nakano [5] and was redefined and generalized by Musielak and Orlicz [7]. Since then, several fixed point and common fixed point theorems in the frame work of modular spaces have been investigated. For more details see ([1] – [4], [6], [8], [10] – [20], [22]). Bhaskar and Lakshmikantham [21] introduced the concept of coupled fixed points and later several authors obtained coupled fixed point and coupled common fixed point theorems in various spaces. Recently Abbas et al. [9] introduced W-compatible mappings in cone metric spaces.

In this paper, combining these concepts, we obtain a coupled common fixed point theorem for Jungck type mappings in modular spaces. We also give an example to illustrate our main theorem.

Now we give some basic definitions on modular spaces.

Definition 1.1 Let X be an arbitrary real or complex vector space. A functional $\rho : X \to [0,\infty)$ is called modular if for any $x, y \in X$, the following conditions hold:

(ρ_1) $\rho(x) = 0$ if and only if $x = 0$,

(ρ_2) $\rho(\lambda x) = |\lambda|\rho(x)$ for every scalar λ with $|\lambda|=1$,

(ρ_3) $\rho(\lambda x + \mu y) \le \rho(x) + \rho(y)$ whenever $\lambda + \mu = 1$ and $\lambda, \mu \ge 0$.

Note that $X_\rho = \{x \in X : \rho(\lambda x) \to 0 \; as \; \lambda \to 0\}$ is called a modular space.

Remark 1.2 *Let* X_ρ be a modular space. Then

(i) $\rho\left(\frac{x}{2}\right) \le \rho(x)$ for all $x \in X_\rho$

Proof. $\rho\left(\frac{x}{2}\right) = \rho\left(\frac{1}{2}x + \frac{1}{2}0\right) \le \rho(x) + \rho(0) = \rho(x)$, from (ρ_3) and (ρ_1).

(ii) $\rho(x+y) \le \rho(2x) + \rho(2y)$ for all $x, y \in X_\rho$

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Proof. $\rho(x+y)=\rho\left(\frac{1}{2}(2x)+\frac{1}{2}(2y)\right)\leq\rho(2x)+\rho(2y)$, from (ρ_3).

Definition 1.3 Let X_ρ be a modular space.

(i) The sequence $\{x_n\}$ in X_ρ is called ρ-convergent to $x\in X_\rho$ if and only if $\lim_{n\to\infty}\rho(x_n-x)=0.$

(ii) ρ-Cauchy if and only if $\lim_{n,m\to\infty}\rho(x_n-x_m)=0$.

(iii) A subset C of X_ρ is called ρ-closed if the ρ-limit of a ρ-convergent sequence of C is still in C.

(iv) A subset C of X_ρ is called ρ-complete if any ρ-Cauchy sequence in C is ρ-convergent and its ρ-limit belongs to C.

(v) ρ is said to satisfy the Δ_2-condition if $\rho(2x_n)\to 0$ whenever $\rho(x_n)\to 0$ as $n\to\infty$.

(vi) we say that ρ has the Fatou property if $\rho(x-y)\leq\lim_{n\to\infty}\inf\rho(x_n-y)$ for all $y\in X_\rho$ whenever $\rho(x_n-x)\to 0$ as $n\to\infty$.

Definition 1.4 Let X_ρ be a modular space. We say that $T:X_\rho\to X_\rho$ is ρ-continuous if $\rho(Tx_n-Tx)\to 0$ whenever $\rho(x_n-x)\to 0$ as $n\to\infty$.

Definition 1.5 Let X_ρ be a modular space and $F:X_\rho\times X_\rho\to X_\rho$ and $g:X_\rho\to X_\rho$. Then the pair (F,g) is said to be ρ-compatible if $\rho(F(gx_n,gy_n)-g(F(x_n,y_n)))\to 0$ and $\rho(F(gy_n,gx_n)-g(F(y_n,x_n,)))\to 0$ as $n\to\infty$ whenever there exist sequences $\{x_n\}$ and $\{y_n\}$ in X_ρ such that $\lim_{n\to\infty}F(x_n,y_n)=\lim_{n\to\infty}gx_n=t$ and $\lim_{n\to\infty}F(y_n,x_n)=\lim_{n\to\infty}gy_n=t'$ for some t and t' in X_ρ.

Definition 1.6 Let X_ρ be a modular space and $F:X_\rho\times X_\rho\to X_\rho$ and $g:X_\rho\to X_\rho$. Then the pair (F,g) is said to be ρ- weakly compatible if $F(gx,gy)=g(F(x,y))$ and $F(gy,gx)=g(F(y,x))$ whenever there exist x and y in X_ρ such that $F(x,y)=gx$ and $F(y,x)=gy$.

Definition 1.7 Let X_ρ be a modular space and $F:X_\rho\times X_\rho\to X_\rho$. Then F is said to be ρ-continuous if

$\lim_{n\to\infty}\rho(F(x_n,y_n)-F(x,y))=0=\lim_{n\to\infty}\rho(F(y_n,x_n)-F(y,x))$ whenever $\lim_{n\to\infty}\rho(x_n-x)=0=\lim_{n\to\infty}\rho(y_n-y)$.

Definition 1.8 ([9]) Let X be a nonempty set. An element $(x,y)\in X\times X$ is called

(i) a coupled coincidence point of $F:X\times X\to X$ and $g:X\to X$ if $gx=F(x,y)$ and $gy=F(y,x)$.

(ii) a common coupled fixed point of $F:X\times X\to X$ and $g:X\to X$ if $x=gx=F(x,y)$ and $y=gy=F(y,x)$.

Kaushik et al.[19]introduced the following α-admissible mapping concept which is a generalization of the concept introduced by Mursaleen et al. [15].

Definition 1.9 *([19])*Let X be a nonempty set and $\alpha : X^2 \times X^2 \to \mathbb{R}^+$ be a function.*Let* $F : X \times X \to X$, $g : X \to X$ be mappings. Then F and g are said to be α -admissible if

$\alpha((gx, gy),(gu, gv)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)),(F(u,v), F(v,u))) \geq 1$ for all $x, y,u,v \in X$.

If $g = I(\text{Identity map})$,then the above definition is the concept of Mursaleen et al.[15].

We say that the pair (F, g) is triangular α -admissible if F and g are α -admissible and if $\alpha((x_1, y_1),(x_2, y_2)) \geq 1,\ \alpha((x_2, y_2),(x_3, y_3)) \geq 1 \Rightarrow \alpha((x_1, y_1),(x_3, y_3)) \geq 1, \forall\ x_1, x_2, x_3, y_1, y_2, y_3 \in X$.

Now we prove our main result.

II. MAIN RESULT

Theorem 2.1 . Let X_ρ be a ρ-complete modular space, where ρ satisfies the Δ_2-condition. Let $F : X_\rho \times X_\rho \to X_\rho$ and $g : X_\rho \to X_\rho$ be mappings satisfying

(2.1.1). $F(X_\rho \times X_\rho) \subseteq g(X_\rho)$,

(2.1.2). $\alpha((gx, gy),(gu, gv))\ \rho(F(x, y) - F(u,v)) \leq \lambda\ M_{x,y}^{u,v}$ for all $x, y,u,v \in X_\rho$, where $\lambda \in (0,1)$ and

$$M_{x,y}^{u,v} = \max \left\{ \begin{array}{l} \rho(gx - gu), \rho(gy - gv), \rho(gx - F(x, y)), \\ \rho(gy - F(y, x)), \rho(gu - F(u,v)), \rho(gv - F(v,u)), \\ \frac{1}{2}\left[\rho\left(\frac{gx - F(u,v)}{2}\right) + \rho\left(\frac{gu - F(x, y)}{2}\right)\right], \\ \frac{1}{2}\left[\rho\left(\frac{gy - F(v,u)}{2}\right) + \rho\left(\frac{gv - F(y, x)}{2}\right)\right] \end{array} \right\}$$

(2.1.3). F and g are ρ -continuous,

(2.1.4). the pair (F, g) is ρ -compatible,

(2.1.5) (a) $\alpha((gx_0, gy_0),(F(x_0, y_0), F(y_0, x_0))) \geq 1$ and

(2.1.5) (b) $\alpha((gy_0, gx_0),(F(y_0, x_0), F(x_0, y_0))) \geq 1$ for some $x_0, y_0 \in X$,

(2.1.6) the pair (F, g) is triangular α -admissible.

Then F and g have a coupled coincidence point in $X \times X$.

Further if we assume that

(2.1.7) $\alpha((gx, gy),(gu, gv)) \geq 1, \alpha((gy, gx),(gv, gu)) \geq 1$

whenever (x, y) and (u,v) are coupled coincidence points of F and g,

then F and g have a unique coupled common fixed point.

Proof. Let $x_0, y_0 \in X_\rho$ satisfying (2.1.5)(a) and (2.1.5)(b).From $(2.1.1)$, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X_ρ such that

$$gx_{n+1} = F(x_n, y_n),\ gy_{n+1} = F(y_n, x_n),\ n = 0,1,2,\cdots (1)$$

From (2.1.5)(a), we have

$$\begin{aligned}
&\alpha((gx_0, gy_0),(F(x_0, y_0), F(y_0, x_0))) \geq 1\\
&\Rightarrow \alpha((gx_0, gy_0),(gx_1, gy_1)) \geq 1, \quad \text{from } (1)\\
&\Rightarrow \alpha((F(x_0, y_0), F(y_0, x_0)),(F(x_1, y_1), F(y_1, x_1))) \geq 1, \quad \text{from } (2.1.6)\\
&\Rightarrow \alpha((gx_1, gy_1),(gx_2, gy_2)) \geq 1, \ \text{from } (1)\\
&\Rightarrow \alpha((F(x_1, y_1), F(y_1, x_1)),(F(x_2, y_2), F(y_2, x_2))) \geq 1, \quad \text{from } (2.1.6)\\
&\Rightarrow \alpha((gx_2, gy_2),(gx_3, gy_3)) \geq 1, \ \text{from } (1).
\end{aligned}$$

Continuing in this way, we have

$$\alpha((gx_n, gy_n),(gx_{n+1}, gy_{n+1})) \geq 1 \ \forall\, n. \qquad (2)$$

Similarly from (2.1.5)(b), we can obtain

$$\alpha((gy_n, gx_n),(gy_{n+1}, gx_{n+1})) \geq 1 \ \forall\, n. \qquad (3)$$

Let $R_n = \max\{\rho(gx_n - gx_{n+1}), \rho(gy_n - gy_{n+1})\}$.

Using (2), consider

$$\begin{aligned}
\rho(gx_{n+1} - gx_{n+2}) &= \rho(F(x_n, y_n) - F(x_{n+1}, y_{n+1}))\\
&\leq \alpha((gx_n, gy_n),(gx_{n+1}, gy_{n+1}))\, \rho(F(x_n, y_n) - F(x_{n+1}, y_{n+1})) \ (4)\\
&\leq \lambda \ M_{x_n, y_n}^{x_{n+1}, y_{n+1}}
\end{aligned}$$

where

$$M_{x_n, y_n}^{x_{n+1}, y_{n+1}} = \max\left\{\begin{array}{l}
\rho(gx_n - gx_{n+1}), \rho(gy_n - gy_{n+1}), \rho(gx_n - gx_{n+1}),\\
\rho(gy_n - gy_{n+1}), \rho(gx_{n+1} - gx_{n+2}), \rho(gy_{n+1} - gy_{n+2}),\\
\frac{1}{2}\left[\rho(\frac{gx_n - gx_{n+2}}{2}) + \rho(\frac{gx_{n+1} - gx_{n+1}}{2})\right],\\
\frac{1}{2}\left[\rho(\frac{gy_n - gy_{n+2}}{2}) + \rho(\frac{gy_{n+1} - gy_{n+1}}{2})\right]
\end{array}\right\}.$$

Consider

$$\begin{aligned}
\rho\left(\frac{gx_n - gx_{n+2}}{2}\right) &= \rho\left(\frac{gx_n - gx_{n+1} + gx_{n+1} - gx_{n+2}}{2}\right)\\
&\leq \rho(gx_n - gx_{n+1}) + \rho(gx_{n+1} - gx_{n+2}), \text{ from } (\rho_3)\\
&\leq 2\max\{\rho(gx_n - gx_{n+1}), \rho(gx_{n+1} - gx_{n+2})\}
\end{aligned}$$

Similary

$$\rho\left(\frac{gy_n - gy_{n+2}}{2}\right) \le 2\max\{\rho(gy_n - gy_{n+1}), \rho(gy_{n+1} - gy_{n+2})\}.$$

Thus $M_{x_n,y_n}^{x_{n+1},y_{n+1}} = \max\{R_n, R_{n+1}\}$.

(4)becomes $\rho(gx_{n+1} - gx_{n+2}) \le \lambda \max\{R_n, R_{n+1}\}$.

Similarly using(2.1.2) and (3),we have $\rho(gy_{n+1} - gy_{n+2}) \le \lambda \max\{R_n, R_{n+1}\}$.

Thus

$$R_{n+1} \le \lambda \max\{R_n, R_{n+1}\} \tag{5}$$

If $\max\{R_n, R_{n+1}\} = R_{n+1}$ for some n then from (5), we have

$R_{n+1} \le \lambda\, R_{n+1}$ which in turn yields that $R_{n+1} = 0$.

By proceeding as in the above we can show that $R_{n+2} = 0, R_{n+3} = 0, \cdots$.

Thus $\{gx_n\}$ and $\{gy_n\}$ are constant Cauchy sequences.

Suppose $\max\{R_n, R_{n+1}\} = R_n$ for all n.

Then from (5), we have $R_{n+1} \le \lambda\, R_n$ for all n.

Thus $R_{n+1} \le \lambda^n\, R_1$ for all n

$\to 0$ as $n \to \infty$.

Thus

$$\rho(gx_n - gx_{n+1}) \to 0 \text{ and } \rho(gy_n - gy_{n+1}) \to 0 \text{ as } n \to \infty. \tag{6}$$

Suppose either $\{gx_n\}$ or $\{gy_n\}$ is not Cauchy.

Then there exists $\varepsilon > 0$ for which we can find sub sequences $\{gx_{m(k)}\}$, $\{gx_{n(k)}\}$, $\{gy_{m(k)}\}$ and $\{gy_{n(k)}\}$ with $m(k) > n(k) > k$ such that for every k

$$\max\{\rho(gx_{m(k)} - gx_{n(k)}), \rho(gy_{m(k)} - gy_{n(k)})\} \ge \varepsilon. \tag{7}$$

Moreover, corresponding to $n(k)$, we can choose $m(k)$ in such a way that it is the smallest integer with $m(k) > n(k)$ and satisfying (7). Then

$$\max\{\rho(2(gx_{m(k)-1} - gx_{n(k)})), \rho(2(gy_{m(k)-1} - gy_{n(k)}))\} < \varepsilon. \tag{8}$$

From (7), we have

$$
\begin{aligned}
\varepsilon &\le \max\{\rho(gx_{m(k)} - gx_{n(k)}), \rho(gy_{m(k)} - gy_{n(k)})\} \\
&= \max\begin{Bmatrix} \rho(gx_{m(k)} - gx_{m(k)-1} + gx_{m(k)-1} - gx_{n(k)}), \\ \rho(gy_{m(k)} - gy_{m(k)-1} + gy_{m(k)-1} - gy_{n(k)}) \end{Bmatrix} \\
&\le \max\begin{Bmatrix} \rho(2(gx_{m(k)} - gx_{m(k)-1})) + \rho(2(gx_{m(k)-1} - gx_{n(k)})), \\ \rho(2(gy_{m(k)} - gy_{m(k)-1})) + \rho(2(gy_{m(k)-1} - gy_{n(k)})) \end{Bmatrix}, \text{ from Remark 1.2}(ii) \\
&\le \max\{\rho(2(gx_{m(k)} - gx_{m(k)-1})), \rho(2(gy_{m(k)} - gy_{m(k)-1}))\} \\
&\quad + \max\{\rho(2(gx_{m(k)-1} - gx_{n(k)})), \rho(2(gy_{m(k)-1} - gy_{n(k)}))\} \\
&< \max\{\rho(2(gx_{m(k)} - gx_{m(k)-1})), \rho(2(gy_{m(k)} - gy_{m(k)-1}))\} + \varepsilon, \text{ from (8).}
\end{aligned}
$$

From (6) and since ρ satisfies the Δ_2 -condition, we have

$$
\lim_{k\to\infty} \max\{\rho(gx_{m(k)} - gx_{n(k)}), \rho(gy_{m(k)} - gy_{n(k)})\} = \varepsilon \tag{9}
$$

Using (2) and triangular property of α , we have

$$
\alpha((gx_{m(k)-1}, gy_{m(k)-1}), (gx_{n(k)-1}, gy_{n(k)-1})) \ge 1.
$$

Now from (2.1.2), we have

$$
\begin{aligned}
&\rho(gx_{m(k)} - gx_{n(k)}) \\
&= \rho(F(x_{m(k)-1}, y_{m(k)-1}) - F(x_{n(k)-1}, y_{n(k)-1})) \\
&\le \alpha((gx_{m(k)-1}, gy_{m(k)-1}), (gx_{n(k)-1}, gy_{n(k)-1}))\, \rho(F(x_{m(k)-1}, y_{m(k)-1}) - F(x_{n(k)-1}, y_{n(k)-1})) \\
&\le \lambda M_{x_{m(k)-1}, y_{m(k)-1}}^{x_{n(k)-1}, y_{n(k)-1}}
\end{aligned} \tag{10}
$$

Where

$$
M_{x_{m(k)-1}, y_{m(k)-1}}^{x_{n(k)-1}, y_{n(k)-1}} = \max\begin{Bmatrix} \rho(gx_{m(k)-1} - gx_{n(k)-1}), \rho(gy_{m(k)-1} - gy_{n(k)-1}), \\ \rho(gx_{m(k)-1} - gx_{m(k)}), \rho(gy_{m(k)-1} - gy_{m(k)}), \\ \rho(gx_{n(k)-1} - gx_{n(k)}), \rho(gy_{n(k)-1} - gy_{n(k)}), \\ \frac{1}{2}\left[\rho\left(\frac{gx_{m(k)-1} - gx_{n(k)}}{2}\right) + \rho\left(\frac{gx_{n(k)-1} - gx_{m(k)}}{2}\right)\right], \\ \frac{1}{2}\left[\rho\left(\frac{gy_{m(k)-1} - gy_{n(k)}}{2}\right) + \rho\left(\frac{gy_{n(k)-1} - gy_{m(k)}}{2}\right)\right] \end{Bmatrix}.
$$

Consider

$$
\begin{aligned}
\rho(gx_{m(k)-1} - gx_{n(k)-1}) &= \rho(gx_{m(k)-1} - gx_{n(k)} + gx_{n(k)} - gx_{n(k)-1}) \\
&\le \rho(2(gx_{m(k)-1} - gx_{n(k)})) + \rho(2(gx_{n(k)} - gx_{n(k)-1})), \text{ from Remark 1.2}(ii) \\
&< \varepsilon + \rho(2(gx_{n(k)} - gx_{n(k)-1})), \text{ from (8)}
\end{aligned}
$$

Similarly $\rho(gy_{m(k)-1} - gy_{n(k)-1}) < \varepsilon + \rho(2(gy_{n(k)} - gy_{n(k)-1}))$.

Consider

$$\rho\left(\frac{gx_{m(k)-1}-gx_{n(k)}}{2}\right)=\rho\left(\frac{gx_{m(k)-1}-gx_{m(k)}+gx_{m(k)}-gx_{n(k)}}{2}\right)$$
$$\leq \rho(gx_{m(k)-1}-gx_{m(k)})+\rho(gx_{m(k)}-gx_{n(k)}), \text{ from } (\rho_3)$$

Similarly

$$\rho\left(\frac{gx_{n(k)-1}-gx_{m(k)}}{2}\right)\leq \rho(gx_{n(k)-1}-gx_{n(k)})+\rho(gx_{n(k)}-gx_{m(k)}),$$

$$\rho\left(\frac{gy_{m(k)-1}-gy_{n(k)}}{2}\right)\leq \rho(gy_{m(k)-1}-gy_{m(k)})+\rho(gy_{m(k)}-gy_{n(k)}),$$

$$\rho\left(\frac{gy_{n(k)-1}-gy_{m(k)}}{2}\right)\leq \rho(gy_{n(k)-1}-gy_{n(k)})+\rho(gy_{n(k)}-gy_{m(k)}).$$

Thus (10) becomes

$$\rho(gx_{m(k)}-gx_{n(k)})\leq \lambda \max\begin{Bmatrix}\varepsilon+\rho(2(gx_{n(k)}-gx_{n(k)-1})),\\ \varepsilon+\rho(2(gy_{n(k)}-gy_{n(k)-1})),\\ \rho(gx_{m(k)-1}-gx_{m(k)}),\rho(gy_{m(k)-1}-gy_{m(k)}),\\ \rho(gx_{n(k)-1}-gx_{n(k)}),\rho(gy_{n(k)-1}-gy_{n(k)}),\\ \frac{1}{2}\begin{bmatrix}\rho(gx_{m(k)-1}-gx_{m(k)})+\rho(gx_{m(k)}-gx_{n(k)})\\ +\rho(gx_{n(k)-1}-gx_{n(k)})+\rho(gx_{n(k)}-gx_{m(k)})\end{bmatrix},\\ \frac{1}{2}\begin{bmatrix}\rho(gy_{m(k)-1}-gy_{m(k)})+\rho(gy_{m(k)}-gy_{n(k)})\\ +\rho(gy_{n(k)-1}-gy_{n(k)})+\rho(gy_{n(k)}-gy_{m(k)})\end{bmatrix}\end{Bmatrix}$$

Similarly we have

$$\rho(gy_{m(k)}-gy_{n(k)})\leq \lambda \max\begin{Bmatrix}\varepsilon+\rho(2(gx_{n(k)}-gx_{n(k)-1})),\\ \varepsilon+\rho(2(gy_{n(k)}-gy_{n(k)-1})),\\ \rho(gx_{m(k)-1}-gx_{m(k)}),\rho(gy_{m(k)-1}-gy_{m(k)}),\\ \rho(gx_{n(k)-1}-gx_{n(k)}),\rho(gy_{n(k)-1}-gy_{n(k)}),\\ \frac{1}{2}\begin{bmatrix}\rho(gx_{m(k)-1}-gx_{m(k)})+\rho(gx_{m(k)}-gx_{n(k)})\\ +\rho(gx_{n(k)-1}-gx_{n(k)})+\rho(gx_{n(k)}-gx_{m(k)})\end{bmatrix},\\ \frac{1}{2}\begin{bmatrix}\rho(gy_{m(k)-1}-gy_{m(k)})+\rho(gy_{m(k)}-gy_{n(k)})\\ +\rho(gy_{n(k)-1}-gy_{n(k)})+\rho(gy_{n(k)}-gy_{m(k)})\end{bmatrix}\end{Bmatrix}$$

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Thus

$$\max\left\{\begin{array}{l}\rho(gx_{m(k)}-gx_{n(k)}),\\ \rho(gy_{m(k)}-gy_{n(k)})\end{array}\right\}\le\lambda\max\left\{\begin{array}{l}\varepsilon+\rho(2(gx_{n(k)}-gx_{n(k)-1})),\\ \varepsilon+\rho(2(gy_{n(k)}-gy_{n(k)-1})),\\ \rho(gx_{m(k)-1}-gx_{m(k)}),\rho(gy_{m(k)-1}-gy_{m(k)}),\\ \rho(gx_{n(k)-1}-gx_{n(k)}),\rho(gy_{n(k)-1}-gy_{n(k)}),\\ \frac{1}{2}\left[\begin{array}{l}\rho(gx_{m(k)-1}-gx_{m(k)})+\rho(gx_{m(k)}-gx_{n(k)})\\ +\rho(gx_{n(k)-1}-gx_{n(k)})+\rho(gx_{n(k)}-gx_{m(k)})\end{array}\right],\\ \frac{1}{2}\left[\begin{array}{l}\rho(gy_{m(k)-1}-gy_{m(k)})+\rho(gy_{m(k)}-gy_{n(k)})\\ +\rho(gy_{n(k)-1}-gy_{n(k)})+\rho(gy_{n(k)}-gy_{m(k)})\end{array}\right]\end{array}\right\}$$

Letting $k\to\infty$ and using (6), (9) and Δ_2 -condition, we get

$\varepsilon\le\lambda\varepsilon$. It is a contradiction.

Hence $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Since X_ρ is ρ -complete space, there exist $x,y\in X_\rho$ such that $gx_n\to x$ and $gy_n\to y$.

Since the pair (F,g) is ρ -compatible, we have $\rho(g(F(x_n,y_n))-F(gx_n,gy_n))\to 0$ and $\rho(g(F(y_n,x_n))-F(gy_n,gx_n))\to 0$.

Since F is ρ -continuous, we have $\rho(F(gx_n,gy_n)-F(x,y))\to 0$ as $n\to\infty$.

Now

$$\rho\left(\frac{gx-F(x,y)}{4}\right)=\rho\left(\frac{\frac{gx-F(gx_n,gy_n)}{2}+\frac{F(gx_n,gy_n)-F(x,y)}{2}}{2}\right)$$

$$\le\rho\left(\frac{gx-F(gx_n,gy_n)}{2}\right)+\rho\left(\frac{F(gx_n,gy_n)-F(x,y)}{2}\right)$$

$$\le\rho(gx-g(F(x_n,y_n))+\rho(g(F(x_n,y_n))-F(gx_n,gy_n))+\rho(F(gx_n,gy_n)-F(x,y)),$$

from (ρ_3) and Remark 1.2(i)

$\to 0$ as $n\to\infty$, since g and F are ρ – continuous and (F, g) *is* ρ – compatible

Thus $gx=F(x,y)$.

Similarly we can show that $gy=F(y,x)$.

Thus (x,y) is a coupled coincidence point of F and g .

Claim: If (u,v) is another coupled coincidence point of F and g , then $gx=gu$ and $gy=gv$.

By (2.1.7), we have $\alpha((gx,gy),(gu,gv))\ge 1$.

Consider

$$\begin{aligned}
\rho(gx-gu) &= \rho(F(x,y)-F(u,v)) \\
&\le \alpha((gx,gy),(gu,gv))\,\rho(F(x,y)-F(u,v)) \\
&\le \lambda \max\left\{\begin{array}{l} \rho(gx-gu), \rho(gy-gv), \rho(gx-F(x,y)), \\ \rho(gy-F(y,x)), \rho(gu-F(u,v)), \rho(gv-F(v,u)), \\ \frac{1}{2}\left[\rho\left(\frac{gx-F(u,v)}{2}\right)+\rho\left(\frac{gu-F(x,y}{2}\right)\right], \\ \frac{1}{2}\left[\rho\left(\frac{gy-F(v,u)}{2}\right)+\rho\left(\frac{gv-F(y,x)}{2}\right)\right] \end{array}\right\} \\
&\le \lambda \max\left\{\begin{array}{l} \rho(gx-gu), \rho(gy-gv), 0,0,0,0, \\ \frac{1}{2}\left[\rho\left(\frac{gx-gu}{2}\right)+\rho\left(\frac{gu-gx}{2}\right)\right], \\ \frac{1}{2}\left[\rho\left(\frac{gy-gv}{2}\right)+\rho\left(\frac{gv-gy}{2}\right)\right] \end{array}\right\} \\
&= \lambda \max\{\rho(gx-gu), \rho(gy-gv)\}, \text{ from Remark 1.2(i)}
\end{aligned}$$

Similarly

$$\rho(gy-gv) \le \lambda \max\{\rho(gx-gu), \rho(gy-gv)\}.$$

Thus

$$\max\left\{\begin{array}{l}\rho(gx-gu), \\ \rho(gy-gv)\end{array}\right\} \le \lambda \max\{\rho(gx-gu), \rho(gy-gv)\}$$

which in turn yields that $gx = gu$ and $gy = gv$. Hence the claim.

Denote $gx = p$ and $gy = q$.

Then $gp = g(gx) = g(F(x,y)) = F(gx,gy) = F(p,q)$, since ρ-compatibility of (F,g) implies ρ-weakly compatibility of (F,g).

Similarly $gq = F(q,p)$. Thus (p,q) is a coupled coincidence point of F and g.

By the claim we have $gx = gp$ and $gy = gq$.

Thus $p = gp$ and $q = gq$.

Hence $F(p,q) = gp = p$ and $F(q,p) = gq = q$.

Hence (p,q) is a coupled common fixed point of F and g.

Suppose (p',q') is another coupled common fixed point of F and g.

Then from (2.1.7), we have $\alpha((gp,gq),(gp',gq')) \ge 1$.

$$\begin{aligned}\rho(p-p') &= \rho(F(p,q)-F(p',q')) \\ &\leq \alpha((gp,gq),(gp',gq'))\,\rho(F(p,q)-F(p',q')) \\ &\leq \lambda \max\left\{\begin{array}{l}\rho(p-p'),\rho(q-q'),\rho(p-p), \\ \rho(q-q),\rho(p'-p'),\rho(q'-q'), \\ \frac{1}{2}\left[\rho\left(\frac{p-p'}{2}\right)+\rho\left(\frac{p'-p}{2}\right)\right], \\ \frac{1}{2}\left[\rho\left(\frac{q-q'}{2}\right)+\rho\left(\frac{q'-q}{2}\right)\right]\end{array}\right\} \\ &= \lambda \max\{\rho(p-p'),\rho(q-q')\}, \text{from Remark } 1.2(i)\end{aligned}$$

Similarly

$$\rho(q-q') \leq \lambda \max\{\rho(p-p'),\rho(q-q')\}.$$

Thus

$$\max\{\rho(p-p'),\rho(q-q')\} \leq \lambda \max\{\rho(p-p'),\rho(q-q')\}$$

which in turn yields that $p=p', q=q'$.

Thus (p,q) is the unique coupled common fixed point of F and g.

Now, we give an example to illustrate Theorem 2.1.

Example 2.2 *Let* $X_\rho = l^1$, where $\rho(x)=\|x\|=\sum|x_i|$ *for* $x \in l^1$. Define $F: l^1 \times l^1 \to l^1$ and $g: l^1 \to l^1$ by $F(x,y)=\left(\frac{x_1+y_1}{12},\frac{x_2+y_2}{12},0,0,\cdots\right)$, $gx=\left(\frac{x_1}{3},\frac{x_2}{3},0,0,\cdots\right)$ for all $x=(x_1,x_2,\cdots)\in l^1$ and $y=(y_1,y_2,\cdots)\in l^1$.

Define $\alpha: X_\rho^2 \times X_\rho^2 \to R^+$ as $\alpha((x,y),(u,v))=\frac{3}{2}$ if $x,y,u,v \in l^1$ with $\|x\|=\|y\|=\|u\|=\|v\|\leq 1$ and

0, otherwise.

Let $x=(x_1,x_2,\cdots), y=(y_1,y_2,\cdots), u=(u_1,u_2,\cdots), v=(v_1,v_2,\cdots)\in l^1$.

Case(a):Suppose $\|x\|=\|y\|=\|u\|=\|v\|\leq 1$.

Consider

$$\begin{aligned}&\alpha((gx,gy),(gu,gv))\,\rho(F(x,y)-F(u,v)) \\ &= \frac{3}{2}\rho\left(\left(\frac{x_1+y_1}{12},\frac{x_2+y_2}{12},0,0,\cdots\right)-\left(\frac{u_1+v_1}{12},\frac{u_2+v_2}{12},0,0,\cdots\right)\right) \\ &= \frac{3}{2}\rho\left(\frac{x_1+y_1-u_1-v_1}{12},\frac{x_2+y_2-u_2-v_2}{12},0,0,\cdots\right) \\ &= \frac{3}{2}\left[\frac{1}{12}|x_1+y_1-u_1-v_1|+\frac{1}{12}|x_2+y_2-u_2-v_2|\right] \\ &\leq \frac{3}{2}\left[\frac{1}{12}(|x_1-u_1|+|y_1-v_1|)+\frac{1}{12}(|x_2-u_2|+|y_2-v_2|)\right] \\ &= \frac{3}{8}\left[\frac{1}{3}(|x_1-u_1|+|x_2-u_2|)+\frac{1}{3}(|y_1-v_1|+|y_2-v_2|)\right]\end{aligned}$$

$$= \frac{3}{8}[\rho(gx-gu)+\rho(gy-gv)]$$
$$\leq \frac{3}{4}\max\{\rho(gx-gu),\rho(gy-gv)\}$$

Case(b): suppose at least one of $\|x\|,\|y\|,\|u\|$ and $\|v\|$ is greater than 1.

Then $\alpha((x,y),(u,v))=0$.

Thus in both cases, (2.1.2) is satisfied with $\lambda=\frac{3}{4}$.One can easily verify the remaining conditions.Clearly $(0,0)$ is the unique coupled common fixed point of F and g.

III. CONCLUSION

By choosing α, F and g in Theorem 2.1, one can obtain several fixed point results in modular spaces.

REFERENCES

1. A.Razani and R.Moradi, “Common fixed point theorems of integral type in modular spaces”, Bulletin of the Iranian Mathematical Society, vol.35 (2), pp. 11 – 24, 2009.
2. C.Mongkolkeha and P.Kumam, “Fixed point and common fixed point theorems for generalized weak contraction mappings of integral type in modular spaces”, Int. J. Math. Sci., vol. 2011, 12 pages, 2011.
3. C.Mongkolkeha and P.Kumam, “Common fixed points for generalized weak contraction mappings in modular spaces”, Sci.Math.Jpn., vol. 75, pp. 69 – 78, 2012.
4. C.Mongkolkeha and P.Kumam, “Some Fixed point results for generalized weak contraction mappings in modular spaces”, Int. J. Anal., vol. 2013 , 6 pages, 2013.
5. H.Nakano, “Modulared semi - ordered linear spaces”, Tokyo Math.Bookser, vol.1, Maruzen, Tokyo (1950).
6. I.Altun, M.Abbas and H.Simsek, “ A fixed point theorem on cone metric spaces with new type contractivity”, Banach J.Math.Anal., vol. 5 (2), pp. 15 – 24, 2011.
7. J.Musielak and W.Orlicz, “On modular spaces”, Stud.Math., vol.18 , pp. 49 – 56, 1959.
8. K.Kuakket and P.Kumam, “ Fixed points of asymptotic pointwise contractions in modular spaces”, Appl.Math.Lett., vol. 24 (11),pp. 1795 – 1798, 2011 .
9. M.Abbas, M.Alikhan and S.Radenovic, “Common coupled fixed point theorems in cone metric space for w - compatible mappings”, Applied Mathematics and Computation, vol. 217 (1), pp. 195 – 202, 2010.
10. M.Abbas, S.Ali and P.Kumam, “Common fixed points in partially ordered modular function spaces”, Journal of inequalities and Applications, vol. 2014, 12 pages, 2014.
11. M.A.Khamsi, “Quasicontraction mapping in modular spaces without Δ_2 - condition”, Fixed Point Theory and Applications, vol. 2008 , 6 pages, 2008.
12. M.A.Khamsi, W.K.Kozolowski and S.Reich, “ Fixed point theory in modular function spaces”, Nonlinear Anal., vol. 14, pp. 935 – 953, 1990.
13. M.A.Kutbi and A.Latif, “ Fixed points of multivalued mappings in modular function spaces”, Fixed Point Theory Appl., vol. 2009, 12 pages, 2009.
14. M.Beygmohammadi and A.Razani, “Two fixed - point theorems for mappings satisfying a general contractive condition of integral type in the modular space”, Int.J.Math. Math.Sci., vol. 2010, 10 pages, 2010.
15. M. Musaleen, S. A. Mohiuddine and R. P.Agarwal, “Coupled fixed point theorems for $\alpha-\psi$ -contractive type mappings in partially ordered metric spaces”,Fixed Point Theory and Applications, vol. 2012,11 pages, 2012.
16. M.Ozturk, M.Abbas and E.Girgin, “Fixed points of mappings satisfying contractive condition of integral type in modular spaces endowed with a graph”, Fixed Point Theory Appl., vol. 2014 (220) , 17 pages, 2014.
17. M.Ozturk, M.Abbas and E.Girgin, “Common fixed point results of a pair of generalized (ψ,φ) - contraction mappings in modular spaces”, Fixed Point Theory and Applications, vol. 2016, 19 pages, 2016.
18. M.Paknazar, M.Eshaghi, J.Cho and S.M.Vaezpour, “A pata - type fixed point theorem in modular spaces with application”, Fixed Point Theory Appl., vol. 2013 (239) , 10 pages, 2013.
19. P.Kaushik,S.Kumar and P.Kumam, “ Coupled coincidence point theorems for $\alpha-\psi$ -contractive type mappings in partially ordered metric spaces”,Fixed Point Theory and Applications, vol. 2013,10 pages, 2013.
20. P.Kumam, “Fixed point theorems for nonexpansive mappings in modular spaces”, Arch.Math., vol. 40, pp. 345 – 353, 2004.
21. T.G. Bhaskar and V. Lakshmikantham, “ Fixed point theorems in partially ordered metric spaces and applications”, Nonlinear Analysis, vol. 65 (7), pp. 1379 – 1393, 2006.
22. T.L.Shateri, “Fixed point in modular spaces with new type contractivity”, International Journal of Applied mathematical research, vol. 3 (4), pp. 587 – 591, 2014.