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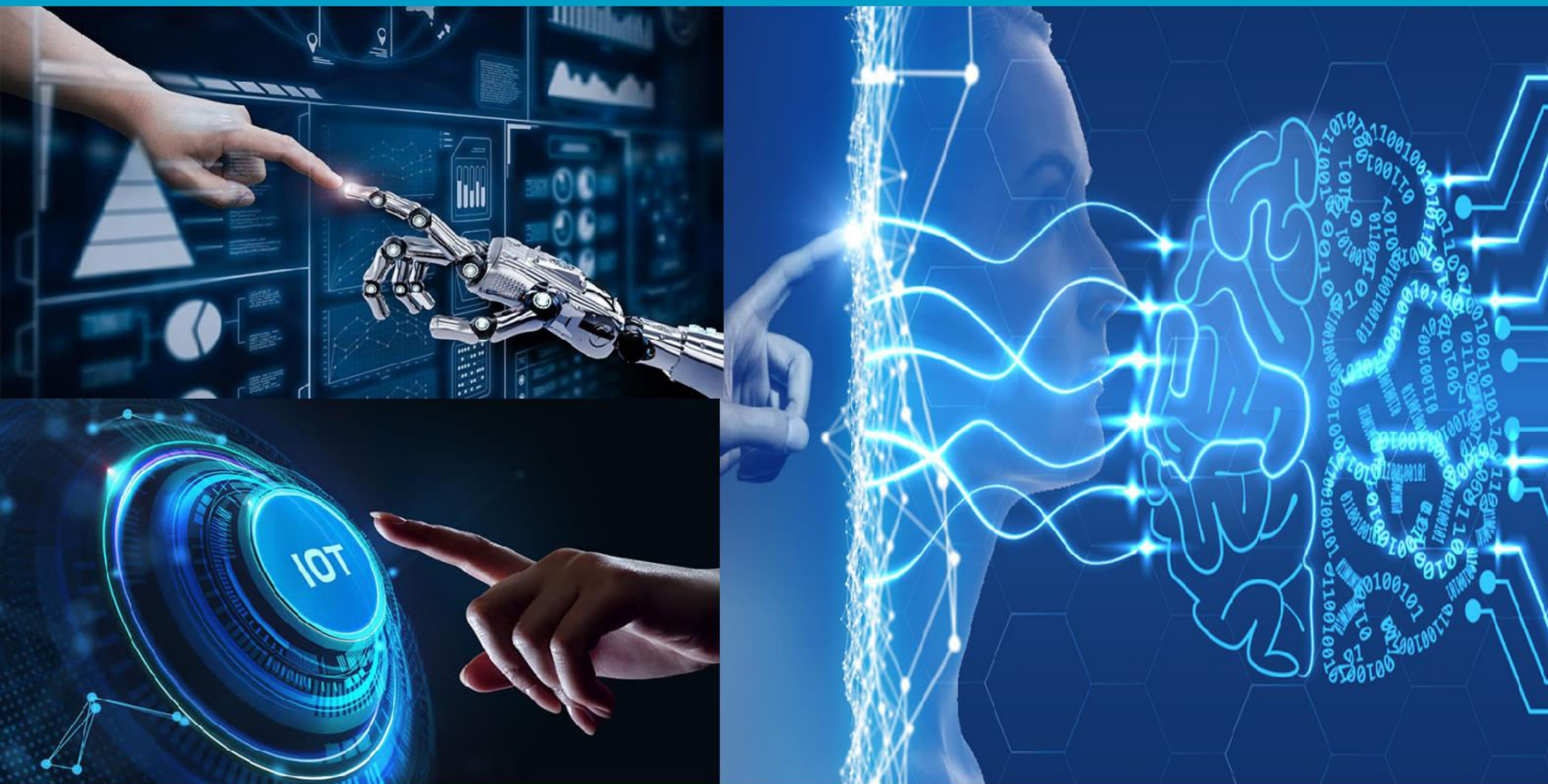

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Algebraic Structure in Discrete Mathematics

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ABSTRACT: The algebraic structure is a type of non-empty set G which is equipped with one or more than one binary operation. Let us assume that * describes the binary operation on non-empty set G. In this case, (G, *) will be known as the algebraic structure. (1, -), (1, +), (N, *) all are algebraic structures.
(R, +, .) is a type of algebraic structure, which is equipped with two operations (+ and .)

KEYWORDS: algebraic, discrete, mathematics, operations, set

I. INTRODUCTION

In the binary operation, binary stands for two. A binary operation is a type of operation that needs two inputs, which are known as the operands. When we perform multiplication, division, addition, or subtraction operations on two numbers, then we will get a number. The two elements of a set are associated with binary operations. The result of these two elements will also be in the same set. So we can say that if we perform a binary operation on a set, then it will perform calculations that combine two elements of the set and generate another element that belongs to the same set.

Let us assume that there is a non-empty set called G. A function f from G × G to G is known as the binary operation on G. So f: G × G → G defines a binary operation on G.

In this example, we will take the two natural numbers or two real numbers and perform binary operations such as addition, multiplication, subtraction, and division on these numbers. The algebraic operation on two natural numbers or real numbers will generate a result. If we get a natural number or real number as a result, then we will consider that binary operation in our set.

Addition:
We will learn about addition, which is a binary operation. Suppose we have two natural numbers(a, b). Now if we add these numbers, then it will generate a natural number as a result. For example: Suppose there are 6 and 8 two natural numbers and the addition of these numbers are[1,2,3]

6 + 8 = 14
Hence, the result 14 is also a natural number. So, we will consider an addition in our set. The same process will be followed for real numbers as well.
+: N + N → N is derived by (a, b) → a + b
+: R + R → R is derived by (a, b) → a + b

Multiplication:
Now we will learn multiplication, which is a binary operation. If we multiply two natural numbers (a, b), then it will generate a natural number as a result. For example: Suppose there are 10 and 5 two natural numbers and the multiplication of these numbers are:
10 * 5 = 50
Hence, the result 50 is also a natural number. So we will consider multiplication in our set. The same process will be followed for real numbers as well.
+: N × N → N is derived by (a, b) → a × b
+: R × R → R is derived by (a, b) → a × b

Subtraction:
Now we will learn subtraction, which is a binary operation. If we subtract two real numbers (a, b), then it will also generate a real number as a result. The same process will not be followed for natural numbers, because if we take two

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natural numbers to perform binary subtraction, then it is not compulsory that it will generate a natural number. For example: Suppose we take two natural numbers 5 and 7 and the subtraction of these numbers are

5 - 7 = -2

Hence, the result is not a natural number. So we will not consider subtraction in our set.

- : R x R → R is derived by (a, b)→ a - b

Division

Now we will learn division, which is a binary operation. If we divide two real numbers (a, b), then it will also generate a real number as a result. The same process will not be followed for natural numbers, because if we take two natural numbers to perform binary division, then it is not compulsory that it will generate a natural number. For example: Suppose we take two natural numbers 10 and 6 and the division of these numbers is

10/6 = 5/3

Hence, the result 5/3 is not a natural number. So we will not consider division in our set.

- : R - R → R is derived by (x, y) → x - y

Properties of Algebraic structure

Commutative: Suppose set G contains a binary operation *. The operation * is called to be commutative in G if it holds the following relation:

x * y= y * x for all x, y in G

Associative: Suppose set G contains a binary operation *. The operation * is called to be associative in G if it holds the following relation:

(x*y)*z = x *(y*z) for all x, y, z in G

Identity: Suppose we have an algebraic system (G, *) and set G contains an element e. That element will be called an identifying element of the set if it contains the following relation:

x * e = e * x = x for all x

Here, element e can be referred to as an identity element of G, and we can also see that it is necessarily unique.

II. DISCUSSION

Inverse: Suppose there is an algebraic system (G, *), and it contains an identity e. We will also assume that the set G contains the elements x and y. The element y will be called an inverse of x if it satisfies the following relation:

x * y = y * x = e

Here, element x can also be referred to as inverse of y, and we can also see that it is necessarily unique. The inverse of x can also be referred to as x^{-1} like this:

$x * x^{-1} = x^{-1} * x = e$

Cancellation Law: Suppose set G contains a binary operation *. The operation * is called to be left cancellation law in G if it holds the following relation:

x * y = x * z implies y = z

It will be called the right cancellation law if it holds the following relation:

y * x = z * x implies y = z

Types of Algebraic structure[4,5,6]

There are various types of algebraic structure, which is described as follows:

- Semigroup
- Monoid
- Group
- Abelian Group

All these algebraic structures have wide application in particular to binary coding and in many other disciplines.

Suppose there is an algebraic structure (G, *), which will be known as semigroup if it satisfies the following condition:

- Closure: The operation * is a closed operation on G that means (a*b) belongs to set G for all a, b ∈
- Associative: The operation * shows an association operation between a, b, and c that means a*(b*c) = (a*b)*c for all a, b, c in G.

Monoid:

A monoid is a semigroup, but it contains an extra identity element (E or e). An algebraic structure (G, *) will be known as a monoid if it satisfies the following condition:

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- Closure: G is closed under operation * that means (a*b) belongs to set G for all a, b ∈
- Associative: Operation * shows an association operation between a, b, and c that means a*(b*c) = (a*b)*c for all a, b, c in G.
- Identity Element: There must be an identity in set G that means a * e = e * a = a for all x.

Group:

A Group is a monoid, but it contains an extra inverse element, which is denoted by 1. An algebraic structure (G, *) will be known as a group if it satisfies the following condition:

- Closure: G is closed under operation * that means (a*b) belongs to set G for all a, b ∈
- Associative: * shows an association operation between a, b, and c that means a*(b*c) = (a*b)*c for all a, b, c in G.
- Identity Element: There must be an identity in set G that means a * e = e * a = a for all a.
- Inverse Element: It contains an inverse element that means $a * a^{-1} = a^{-1} * a = e$ for a ∈

Abelian Group

An abelian group is a group, but it contains commutative law. An algebraic structure (G, *) will be known as an abelian group if it satisfies the following condition:

- Closure: G is closed under operation * that means (a*b) belongs to set G for all a, b ∈
- Associative: * shows an association operation between a, b, and c that means a*(b*c) = (a*b)*c for all a, b, c in G.
- Identity Element: There must be an identity in set G that means a * e = e * a = a for all a.
- Inverse Element: It contains an inverse element that means $a * a^{-1} = a^{-1} * a = e$ for a ∈
- Commutative Law: There will be a commutative law such that a * b = b * a such that a, b belongs to G.

III. RESULTS

In mathematics, an algebraic structure consists of a nonempty set A (called the underlying set, carrier set or domain), a collection of operations on A (typically binary operations such as addition and multiplication), and a finite set of identities, known as axioms, that these operations must satisfy.

An algebraic structure may be based on other algebraic structures with operations and axioms involving several structures. For instance, a vector space involves a second structure called a field, and an operation called scalar multiplication between elements of the field (called scalars), and elements of the vector space (called vectors).

Abstract algebra is the name that is commonly given to the study of algebraic structures. The general theory of algebraic structures has been formalized in universal algebra. Category theory is another formalization that includes also other mathematical structures and functions between structures of the same type (homomorphisms).[7,8,9]

In universal algebra, an algebraic structure is called an algebra;[1] this term may be ambiguous, since, in other contexts, an algebra is an algebraic structure that is a vector space over a field or a module over a commutative ring.

The collection of all structures of a given type (same operations and same laws) is called a variety in universal algebra; this term is also used with a completely different meaning in algebraic geometry, as an abbreviation of algebraic variety. In category theory, the collection of all structures of a given type and homomorphisms between them form a concrete category.

Simple structures: no binary operation:

- Set: a degenerate algebraic structure S having no operations.
- Group-like structures: one binary operation. The binary operation can be indicated by any symbol, or with no symbol (juxtaposition) as is done for ordinary multiplication of real numbers.
- Group: a monoid with a unary operation (inverse), giving rise to inverse elements.
- Abelian group: a group whose binary operation is commutative.
- Ring-like structures or Ringoids: two binary operations, often called addition and multiplication, with multiplication distributing over addition.
- Ring: a semiring whose additive monoid is an abelian group.
- Division ring: a nontrivial ring in which division by nonzero elements is defined.
- Commutative ring: a ring in which the multiplication operation is commutative.

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- Field: a commutative division ring (i.e. a commutative ring which contains a multiplicative inverse for every nonzero element).
- Lattice structures: two or more binary operations, including operations called meet and join, connected by the absorption law.[2]
- Complete lattice: a lattice in which arbitrary meet and joins exist.
- Bounded lattice: a lattice with a greatest element and least element.
- Distributive lattice: a lattice in which each of meet and join distributes over the other. A power set under union and intersection forms a distributive lattice.
- Boolean algebra: a complemented distributive lattice. Either of meet or join can be defined in terms of the other and complementation.
- Two sets with operations
- Module: an abelian group M and a ring R acting as operators on M. The members of R are sometimes called scalars, and the binary operation of scalar multiplication is a function R × M → M, which satisfies several axioms. Counting the ring operations these systems have at least three operations.
- Vector space: a module where the ring R is a division ring or field.
- Algebra over a field: a module over a field, which also carries a multiplication operation that is compatible with the module structure. This includes distributivity over addition and linearity with respect to multiplication.
- Inner product space: an F vector space V with a definite bilinear form V × V → F.
- Hybrid structures
- Algebraic structures can also coexist with added structure of non-algebraic nature, such as partial order or a topology. The added structure must be compatible, in some sense, with the algebraic structure.[10,11,12]
- Topological group: a group with a topology compatible with the group operation.
- Lie group: a topological group with a compatible smooth manifold structure.
- Ordered groups, ordered rings and ordered fields: each type of structure with a compatible partial order.
- Archimedean group: a linearly ordered group for which the Archimedean property holds.
- Topological vector space: a vector space whose M has a compatible topology.
- Normed vector space: a vector space with a compatible norm. If such a space is complete (as a metric space) then it is called a Banach space.
- Hilbert space: an inner product space over the real or complex numbers whose inner product gives rise to a Banach space structure.
- Vertex operator algebra
- Von Neumann algebra: a *-algebra of operators on a Hilbert space equipped with the weak operator topology.

Universal algebra

Algebraic structures are defined through different configurations of axioms. Universal algebra abstractly studies such objects. One major dichotomy is between structures that are axiomatized entirely by identities and structures that are not. If all axioms defining a class of algebras are identities, then this class is a variety (not to be confused with algebraic varieties of algebraic geometry).

Identities are equations formulated using only the operations the structure allows, and variables that are tacitly universally quantified over the relevant universe. Identities contain no connectives, existentially quantified variables, or relations of any kind other than the allowed operations. The study of varieties is an important part of universal algebra. An algebraic structure in a variety may be understood as the quotient algebra of term algebra (also called "absolutely free algebra") divided by the equivalence relations generated by a set of identities. So, a collection of functions with given signatures generate a free algebra, the term algebra T. Given a set of equational identities (the axioms), one may consider their symmetric, transitive closure E. The quotient algebra T/E is then the algebraic structure or variety. Thus, for example, groups have a signature containing two operators: the multiplication operator m, taking two arguments, and the inverse operator i, taking one argument, and the identity element e, a constant, which may be considered an operator that takes zero arguments. Given a (countable) set of variables x, y, z, etc. the term algebra is the collection of all possible terms involving m, i, e and the variables; so for example, m(i(x), m(x, m(y,e))) would be an element of the term algebra. One of the axioms defining a group is the identity m(x, i(x)) = e; another is m(x,e) = x. The axioms can be represented as trees. These equations induce equivalence classes on the free algebra; the quotient algebra then has the algebraic structure of a group.
Some structures do not form varieties, because either:

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1. It is necessary that $0 \neq 1$, 0 being the additive identity element and 1 being a multiplicative identity element, but this is a nonidentity;
2. Structures such as fields have some axioms that hold only for nonzero members of S. For an algebraic structure to be a variety, its operations must be defined for all members of S; there can be no partial operations.

Structures whose axioms unavoidably include nonidentities are among the most important ones in mathematics, e.g., fields and division rings.

Category theory

Category theory is another tool for studying algebraic structures (see, for example, Mac Lane 1998). A category is a collection of objects with associated morphisms.[16,17,18] Every algebraic structure has its own notion of homomorphism, namely any function compatible with the operation(s) defining the structure. In this way, every algebraic structure gives rise to a category. For example, the category of groups has all groups as objects and all group homomorphisms as morphisms. This concrete category may be seen as a category of sets with added category-theoretic structure. Likewise, the category of topological groups (whose morphisms are the continuous group homomorphisms) is a category of topological spaces with extra structure. A forgetful functor between categories of algebraic structures "forgets" a part of a structure.[13,14,15]

IV. CONCLUSION

There are various concepts in category theory that try to capture the algebraic character of a context, for instance[19]

- algebraic category
- essentially algebraic category
- presentable category
- locally presentable category
- monadic functors and categories
- universal property.[20]

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