ISSN: 2395-7639

International Journal of Multidisciplinary Research in Science, Engineering, Technology & Management (IJMRSETM)

(A Monthly, Peer Reviewed Online Journal)

Visit: www.ijmrsetm.com

Volume 3, Issue 8, August 2016

Study of a Smooth Dynamic Punch on the Boundary of a Transversely Isotropic Half-Plane

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ABSTRACT: This paper deals with the study of a smooth dynamic punch situated along the boundary of a transversely isotropic half-plane. The complex variable technique has been adopted to obtain closed form solution of the elastodynamic punch problem. The stress and displacement fields are expressed in terms of two analytic functions in appropriate complex domains. The boundary value problem has then been solved using Riemann-Hilbert technique. Expressions for the contact pressure and resultant moment of external forces restraining the stamp are obtained in closed form.

KEYWORDS: Transversely isotropic elastic medium, Punch, Riemann-Hilbert technique, contact pressure, resultant moment of external forces.

I. INTRODUCTION

Punch problems belong to a certain class called contact problems within the theory of elasticity. Contact problems are related to the name of Hertz, who first in 1882 successfully treated a static contact problem. Discussions on punch problems in isotropic media are abundant. Detailed reference in this direction may be found in the book of Gladwell [1980]. Contact problems in anisotropic meida under dynamical conditions are comparatively difficult due to their inherent mathematical complexities. Some significant works on contact problem in anisotropic media by several authors are referred to as Willis [1966], Beddings and Willis [1973], Miller [1986] and Fabrikant [1986]. The punch problem is of great importance in solid mechanics for its multiple technical applications including ballistic impact, explosives, metal forming and manufacturing operations such as punching and blanking. While the quasi-static punch problem is a well-studied field in contact mechanics (Johnson [1987], Gladwell [1980]), there is a little work on the dynamic case. In the dynamic case the punch approaches the material with a certain velocity, hence wave propagation is involved complicating the mathematical analysis. Brock [1983, 1996] considered the problem of rapid indentation of an isotropic half-plane by a smooth, flat and rigid semi-infinite punch. Punch geometries other than flat, have been considered even for general anisotropic materials for example parabolic and wedge shaped punches, (Willis [1973]; Georgiadis and Brock [1995]). In those problems the contact area grows at constant speed and self-similar feature of the dynamic fields is exploited using the general methodology derived by Willis [1973] for such kind of problems. The dynamic, flat finite punch problem in isotropic materials was considered by Roessig and Manson [1998]. They solved an equivalent problem where the rigid punch is replaced by a compressive wave impinging two semi-infinite external cracks. In current years, very few works have been done on punch problems among which Guler [2014] studied closed form solution of the two-dimensional sliding frictional contact problem for an orthotropic medium. Bedoidze and Pozharskii [2014] studied the interaction of punches on a transversely isotropic half-space. Srefan Rasche and Meinhard Kuna [2015] studied improved small punch testing and parameter identification of ductile to brittle materials. Zhou and Lee [2014] studied dynamic behavior of a moving frictional punch over the surface of anisotropic materials.

The present work deals with the study of elastodynamic problem of a smooth punch moving with constant speed c in the x-direction and situated along the boundary of a transversely isotropic half-plane. By representing the stress and displacement components in terms of two holomorphic functions defined in appropriate complex domains, the problem is solved by Riemann-Hilbert technique. Expressions for contact pressure and resultant moment of external forces restraining the stamp are obtained in closed form.

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II. PROBLEM FORMULATION

2.1 Basic equations:

Let a transversely isotropic medium be referred to as Cartesian coordinate system. The stress-strain relations in matrix form are

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{31}\\ \sigma_{32}\\ \sigma_{12}\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0\\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2}\end{bmatrix}\begin{bmatrix}e_{11}\\ e_{22}\\ e_{33}\\ 2e_{31}\\ 2e_{32}\\ 2e_{12}\end{bmatrix}$$

where σ_{ij} is the stress tensor, C_{ij} are elastic constants, e_{ij} are the components of strain tensor.

The strain-displacement relations are

$$e_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\quad,\quad 1\le i,j\le 3$$

where $(u_1=u,u_2=v,u_3=w)$ are the displacement components and $(x_1=x,x_2=y,x_3=z)$ are cartesian coordinates.

Considering the problem to be restricted to motion in the xy-plane, the displacement equations of motion are

$$\left.\begin{aligned}&C_{11}\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}(C_{11}+C_{12})\frac{\partial^2 v}{\partial x\partial y}+\frac{1}{2}(C_{11}-C_{12})\frac{\partial^2 u}{\partial y^2}=\rho\frac{\partial^2 u}{\partial t^2}\\ &\frac{1}{2}(C_{11}-C_{12})\frac{\partial^2 v}{\partial x^2}+\frac{1}{2}(C_{11}+C_{12})\frac{\partial^2 u}{\partial x\partial y}+C_{11}\frac{\partial^2 v}{\partial y^2}=\rho\frac{\partial^2 v}{\partial t^2}\end{aligned}\right\}\qquad(2.1.1)$$

The stress-displacement relations are given as

$$\left.\begin{aligned}&\sigma_{xx}=C_{11}\frac{\partial u}{\partial x}+C_{12}\frac{\partial v}{\partial y}\\ &\sigma_{yy}=C_{12}\frac{\partial u}{\partial x}+C_{11}\frac{\partial v}{\partial y}\\ &\sigma_{xy}=\frac{1}{2}(C_{11}-C_{12})\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\end{aligned}\right\}\qquad(2.1.2)$$

2.2 Formulation of the Problem:

By setting Galilean transformation

$$X=x-ct\ ,\ Y=y\ ,\ t=t\ ,\qquad(2.2.1)$$

the system of equations (2.1.1) reduces to

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$$I\frac{\partial \phi}{\partial X}+A\frac{\partial \phi}{\partial Y}=0, \tag{2.2.2}$$

where c is a constant, $u=u(X,Y)$, $v=v(X,Y)$, I is 4x4 identity matrix and A is a 4x4 matrix given by

$$A=\begin{pmatrix} 0 & \alpha & 2\beta & 0 \\ -1 & 0 & 0 & 0 \\ 2\beta_1 & 0 & 0 & \alpha_1 \\ 0 & 0 & -1 & 0 \end{pmatrix},\ \phi=\left\{\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right\}^T ,$$

in which

$$2\beta=\frac{(C_{11}+C_{12})/2}{C_{11}(1-M_1{}^2)},\ 2\beta_1=\frac{(C_{11}+C_{12})}{(C_{11}-C_{12})(1-M_2{}^2)},\ \alpha=\frac{(C_{11}-C_{12})/2}{C_{11}(1-M_1{}^2)},\ \alpha_1=\frac{2C_{11}}{(C_{11}-C_{12})(1-M_2{}^2)},$$

$M_j={}^{c}\!/_{v_j}$, $(j=1,2)$ are Mach numbers with $v_1=\left(\frac{C_{11}}{\rho}\right)^{1/2}$, $v_2=\left(\frac{((C_{11}-C_{12})/2)}{\rho}\right)^{1/2}$.

The Eigen values of matrix A are given by $\pm ip$, $\pm iq$

where $p=\left[a_1-\left(a_1{}^2-a_2\right)^{1/2}\right]^{1/2}$, $q=\left[a_1+\left(a_1{}^2-a_2\right)^{1/2}\right]^{1/2}$

in which $2a_1=\alpha+\alpha_1-4\beta\beta_1$ and $a_2=\alpha\alpha_1$.

We now consider the transformation $\phi(X,Y)=T\psi(X,Y)$, (2.2.3)

$$\text{where } T=\begin{bmatrix} 0 & \frac{2\beta p^2}{\alpha-p^2} & 0 & \frac{2\beta p^2}{\alpha-p^2} \\ \frac{2\beta p^2}{\alpha-p^2} & 0 & \frac{2\beta p^2}{\alpha-p^2} & 0 \\ -p & 0 & -q & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \tag{2.2.4}$$

It can be shown that $\psi_j(j=1,2)$ and $\psi_k(k=3,4)$ are pairs of conjugate harmonic functions into $Z_1=X+iY_1$ and $Z_2=X+iY_2$ plane respectively with $Y_1=Y/p$, $Y_2=Y/q$.

By setting $\Omega_1(z_1)=\psi_1+i\psi_2$, $\Omega_2(z_2)=\psi_3+i\psi_4$, the stress field can be expressed as

$$\sigma_{XX}=\frac{(C_{11}-C_{12})/2}{\alpha}\operatorname{Im}[l_1\Omega_1(z_1)+l_2\Omega_2(z_2)], \tag{2.2.5a}$$

$$\sigma_{YY}=\frac{(C_{11}-C_{12})}{2}\operatorname{Im}[p^2l_3\Omega_1(z_1)+q^2l_4\Omega_2(z_2)], \tag{2.2.5b}$$

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$$\sigma_{XY} = \frac{(C_{11}-C_{12})}{2}\mathrm{Re}[pl_5\Omega_1(z_1)+ql_6\Omega_2(z_2)], \tag{2.2.5c}$$

where

$$l_1 = \frac{2\beta p^2}{(\alpha-p^2)(1-M_1{}^2)}+(2\beta-\alpha)\ ,\ l_2 = \frac{2\beta q^2}{(\alpha-q^2)(1-M_1{}^2)}+(2\beta-\alpha), l_3 = (1-M_2{}^2)-\frac{2\beta}{(\alpha-p^2)}\ ,$$

$$l_4 = (1-M_2{}^2)-\frac{2\beta}{(\alpha-q^2)},\ l_5 = -(l_3+M_2{}^2)\ ,\ l_6 = -(l_4+M_2{}^2)\ . \tag{2.2.6}$$

The corresponding displacement components are

$$\left.\begin{aligned} u(X,Y) &= \mathrm{Im}\left[\frac{2\beta p^2}{\alpha-p^2}w_1(z_1)+\frac{2\beta q^2}{\alpha-q^2}w_2(z_2)\right], \\ v(X,Y) &= -\mathrm{Re}[pw_1(z_1)+qw_2(z_2)] \end{aligned}\right\} \tag{2.2.7}$$

where

$$\left.\begin{aligned} w_1(z_1) &= \int\Omega_1(z_1)dX = \frac{i}{p}\int\Omega_1(z_1)dY\ , \\ w_2(z_2) &= \int\Omega_2(z_2)dX = \frac{i}{q}\int\Omega_2(z_2)dY\ . \end{aligned}\right\} \tag{2.2.8}$$

2.3 Boundary Conditions:

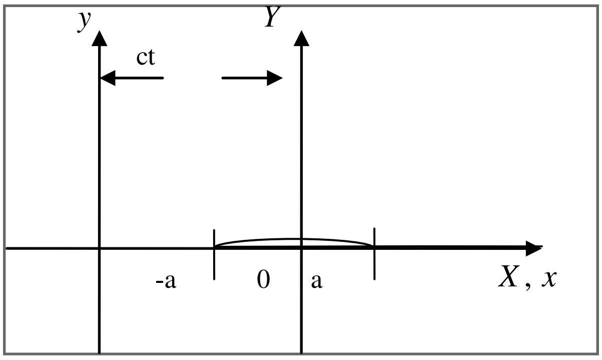

Fig.1: Dynamic punch on elastic half-plane situated on a line segment.

We consider a semi-infinite transversely isotropic plane ($y \geq 0$) in which there is a punch defined by $|X| \leq a$, $Y = 0$ and moving with constant speed c along the positive x-axis as depicted in figure 1. The boundary conditions of the problem are

$$v(X,0) = f(X)\ ,\qquad |X| \leq a \tag{2.3.1}$$

$$\sigma_{YY}(X,0) = 0\ ,\qquad |X| > a \tag{2.3.2}$$

$$\sigma_{XY}(X,0)=0 \quad , \qquad |X|<\infty \tag{2.3.3}$$

$$\sigma_{XX}, \sigma_{YY}, \sigma_{XY}=0 \quad , \quad |Z_j|\to\infty \quad . \tag{2.3.4}$$

III. SOLUTION OF THE PROBLEM

Setting, $\Lambda_1(z_1)=\dfrac{ip\Delta}{l_6}\Omega_1(z_1)$ and $\Lambda_2(z_2)=\dfrac{-iq\Delta}{l_5}\Omega_2(z_2)$, the stress components in (2.2.5a)-(2.2.5c) may be written as follows:

$$\left.\begin{aligned}
\sigma_{XX} &= \frac{(C_{11}-C_{12})/2}{\alpha\Delta}\mathrm{Re}\left[\frac{l_2 l_5}{q}\Lambda_2(z_2)-\frac{l_1 l_6}{p}\Lambda_1(z_1)\right]\\
\sigma_{YY} &= \frac{(C_{11}-C_{12})}{2\Delta}\mathrm{Re}\left[q l_4 l_5\Lambda_2(z_2)-p l_3 l_6\Lambda_1(z_1)\right]\\
\sigma_{XY} &= \frac{(C_{11}-C_{12})}{2\Delta}l_5 l_6\,\mathrm{Im}\left[\Lambda_1(z_1)-\Lambda_2(z_2)\right]
\end{aligned}\right\} \tag{3.1}$$

where $\Delta = p l_3 l_6 - q l_4 l_5$.

Boundary condition (2.3.3) in conjunction with σ_{XY} in (3.1) lead to

$$\Lambda_1(X)=\Lambda_2(X) \quad , \quad |X|<\infty \quad . \tag{3.2}$$

Using the symmetry property $\Lambda_j(z_j)=\overline{\Lambda_j}(z_j)$, boundary conditions (2.3.1) and (2.3.2) by virtue of (3.2) on $Y=0$, lead to

$$\mathrm{Im}\,\Lambda_j(z_j)=\frac{\Delta f'(X)}{(l_5-l_6)} \quad , \quad |X|\le a \tag{3.3}$$

$$\mathrm{Re}\,\Lambda_j(z_j)=0 \quad , \quad |X|>a\,. \tag{3.4}$$

Setting, $\Lambda_j(z_j)=\dfrac{\phi_j(z_j)}{\sqrt{z_j{}^2-a^2}}$, (3.5)

equations (3.3) and (3.4) under (3.5) reduces to the standard Riemann-Hilbert problem,

$$\left.\begin{aligned}
\phi_j(X+i0)-\phi_j(X-i0) &= \frac{-2\Delta\sqrt{a^2-X^2}\,f'(X)}{l_5-l_6} \quad , \quad |X|\le a\\
\phi_j(X+i0)-\phi_j(X-i0) &= 0 \quad , \quad |X|>a\,.
\end{aligned}\right\} \tag{3.6}$$

Following the method of Gakov [1966], solution of the problem (3.6) is given by

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$$\phi_j(z_j) = \frac{-\Delta}{\pi i(l_5 - l_6)} I_j + C \tag{3.7}$$

where $$I_j = \int_{-a}^{a} \frac{f'(t)\sqrt{a^2 - t^2}}{t - z_j} dt \quad . \tag{3.8}$$

Let us consider the particular case where, $f'(X) = f_0$ (a constant) then $I_j = -f_0 \pi \left\{ z_j - \sqrt{z_j^{\ 2} - a^2} \right\} + C$, where C is an arbitrary constant to be determined.

Now from (3.7), $\phi_j(z_j) = \frac{\Delta f_0}{i(l_5 - l_6)} \left(z_j - \sqrt{z_j^{\ 2} - a^2} \right) + C$.

Therefore, $\Lambda_j(z_j) = \frac{\phi_j(z_j)}{\sqrt{z_j^{\ 2} - a^2}} = \frac{\Delta f_0}{i(l_5 - l_6)} \left[\frac{z_j}{\sqrt{z_j^{\ 2} - a^2}} - 1 \right] + \frac{C}{\sqrt{z_j^{\ 2} - a^2}}$.

Using boundary condition (2.3.4), $\Lambda_j(z_j) \to 0$ as $|z_j| \to \infty$ whence $C = 0$, therefore

$$\Lambda_j(z_j) = \frac{\Delta f_0}{i(l_5 - l_6)} \left[\frac{z_j}{\sqrt{z_j^{\ 2} - a^2}} - 1 \right] \tag{3.9}$$

Therefore, the stress components can be determined from (3.1).

The expression for the contact pressure is given by

$$P(X) = \sigma_{YY}(X, 0) = \frac{(C_{11} - C_{12})\Delta f_0}{2(l_5 - l_6)} \frac{X}{\sqrt{a^2 - X^2}} . \tag{3.10}$$

Resultant moment of external forces restraining the punch is given by

$$M = -\int_{-a}^{a} x P(x) dx = \frac{\pi a^2}{2} \frac{\Delta f_0 (C_{11} - C_{12})}{2(l_5 - l_6)} . \tag{3.11}$$

Now, considering the same problem in an isotropic half-plane and then by employing the same method of solution, the expression for the contact pressure and the resultant moment of external forces restraining the stamp are obtained in closed form.

IV. NUMERICAL RESULTS

As an illustration, the expression for the resultant moment of external forces restraining the stamp is computed numerically for various speed c and for different transversely isotropic material. The values of elastic constant have been taken from Lubarda and Chen [2008] and Dinesh et.al. [2012]. Numerical calculations are carried out by taking $a = 1$ and $f_0 = 1$ in the expression (3.11).

The values of resultant moment of external forces restraining the stamp of transversely isotropic material viz. aluminium oxide, graphite and monocrystalline zinc for various crack speed are given in Table I. Similar results are also obtained for isotropic materials viz. alluminium and copper.

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Table I: Values of resultant moment of different transversely isotropic material for various crack speed.

M_2	M		
	Alluminium Oxide	Graphite	Monocrystalline Zinc
0.1	-739	-257.92	-110.5
0.2	-720.1	-251.6	-107.7
0.3	-687.8	-240.79	-102.9
0.4	-640.5	-225.01	-95.82
0.5	-575.9	-203.45	-86.19
0.6	-489.6	-174.72	-73.34
0.7	-374	-136.3	-56.12
0.8	-213	-82.95	-32.14

The result obtained in Table I are then plotted graphically. Figures 2a, 2b and 2c shows the variation of resultant moment of external forces with various values of crack speed for transversely isotropic materials viz. alluminium oxide, graphite and monocrystalline zinc.

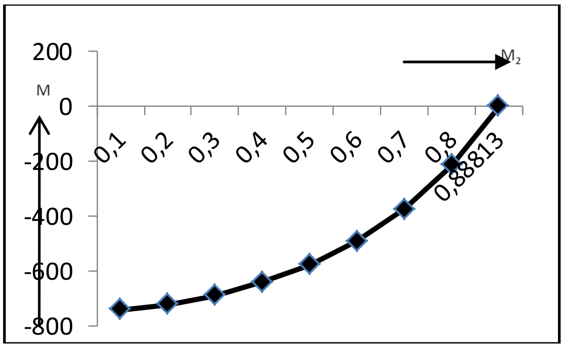

Fig. 2a: M versus M_2 for Aluminium Oxide.

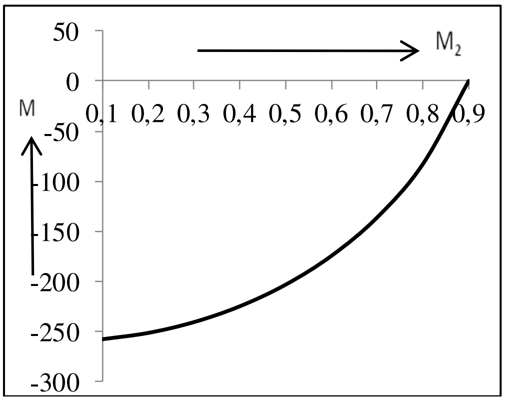

Fig. 2b: M versus M_2 for Graphite.

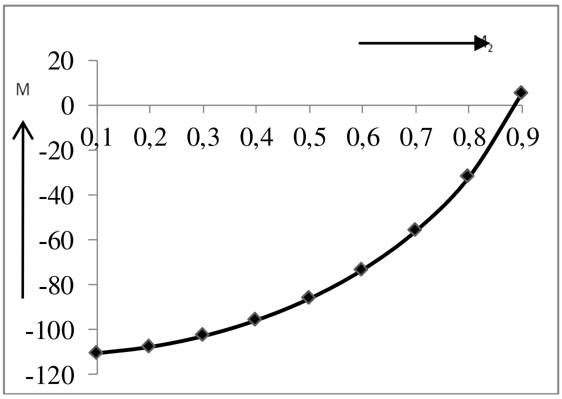

Fig. 2c: M versus M_2 for Monocrystalline zinc.

Now for isotropic materials, the variations of resultant moment of external forces with crack speed are depicted in figures 3a and 3b.

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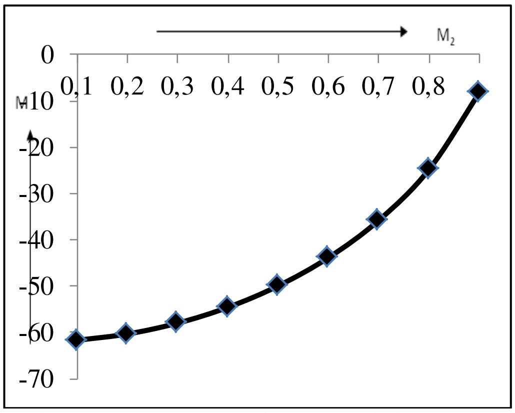

Fig.3a: M versus M_2 for Alluminium.

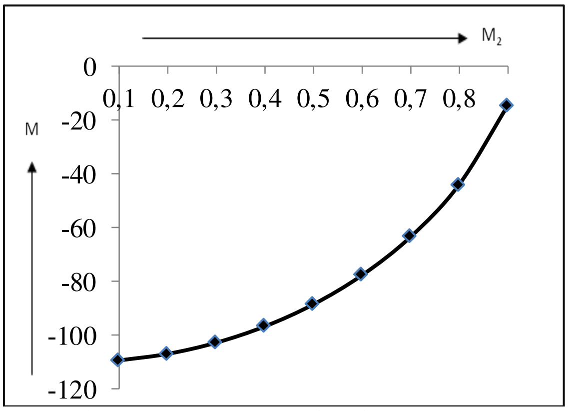

Fig. 3b: M versus M_2 for Copper.

V. CONCLUSION

In this work, elastodynamic response of a smooth moving punch has been investigated. Resultant moment of external forces restraining the punch is obtained to study the numerical results. From figures 2-6, it is to be observed that resultant moment of external forces M , increases as crack speed increases. Nature of curves is same for both isotropic and transversely isotropic material.

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