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The Mathematics Behind Differential Calculus

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Abstract: Differential Calculus is that branch of mathematical analysis, devised by Isaac Newton and G.W. Leibniz It is concerned with the problem of finding the rate of change of a function with respect to the variable on which it depends. Thus, it involves calculating derivatives and using them to solve problems involving nonconstant rates of change. Typical applications include finding maximum and minimum values of functions in order to solve practical problems in optimization.

KEYWORDS: calculus, derivative, differentiation, L'Hôpital's rule, quotient rule.

I.INTRODUCTION

1.1 Fundamental theorem of calculus

Fundamental theorem of calculus is the basic principle of calculus. It relates the derivative to the integral and provides the principal method for evaluating definite integrals (*see*differential calculus; integral calculus). In brief, it states that any function that is continuous (*see*continuity) over an interval has an antiderivative (a function whose rate of change, or derivative, equals the function) on that interval. Further, the definite integral of such a function over an interval $a<x<b$ is the difference $F(b) - F(a)$, where F is an antiderivative of the function. This particularly elegant theorem shows the inverse function relationship of the derivative and the integral and serves as the backbone of the physical sciences. It was articulated independently by Isaac Newton and Gottfried Wilhelm Leibniz.

1.2 Derivative

Derivative, in Mathematics, the rate of change of a function with respect to a variable. Derivatives are fundamental to the solution of problems in calculus and differential equations. In general, scientists observe changing systems (dynamical systems) to obtain the rate of change of some variable of interest, incorporate this information into some differential equation, and use integration techniques to obtain a function that can be used to predict the behaviour of the original system under diverse conditions.

Geometrically, the derivative of a function can be interpreted as the slope of the graph of the function or, more precisely, as the slope of the tangentline at a point. Its calculation, in fact, derives from the slope formula for a straight line, except that a limiting process must be used for curves. The slope is often expressed as the "rise" over the "run," or, in Cartesian terms, the ratio of the change in y to the change in x. For the straight line shown in the figure, the formula for the slope is $(y_1 - y_0)/(x_1 - x_0)$. Another way to express this formula is $[f(x_0 + h) - f(x_0)]/h$, if h is used for $x_1 - x_0$ and $f(x)$ for y. This change in notation is useful for advancing from the idea of the slope of a line to the more general concept of the derivative of a function.

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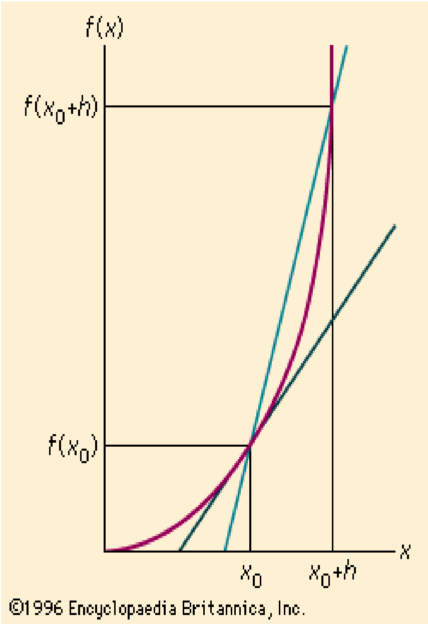
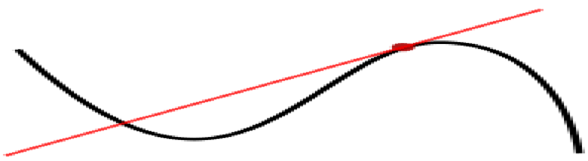


Figure 1 showing the slope of a curve

For a curve, this ratio depends on where the points are chosen, reflecting the fact that curves do not have a constant slope. To find the slope at a desired point, the choice of the second point needed to calculate the ratio represents a difficulty because, in general, the ratio will represent only an average slope between the points, rather than the actual slope at either point (*see* figure). To get around this difficulty, a limiting process is used whereby the second point is not fixed but specified by a variable, as h in the ratio for the straight line above. Finding the limit in this case is a process of finding a number that the ratio approaches as h approaches 0, so that the limiting ratio will represent the actual slope at the given point. Some manipulations must be done on the quotient $[f(x_0 + h) - f(x_0)]/h$ so that it can be rewritten in a form in which the limit as h approaches 0 can be seen more directly.Consider, for example, the parabola given by x^2. In finding the derivative of x^2 when x is 2, the quotient is $[(2 + h)^2 - 2^2]/h$. By expanding the numerator, the quotient becomes $(4 + 4h + h^2 - 4)/h = (4h + h^2)/h$. Both numerator and denominator still approach 0, but if h is not actually zero but only very close to it, then h can be divided out, giving $4 + h$, which is easily seen to approach 4 as h approaches 0.

To sum up, the derivative of $f(x)$ at x_0, written as $f'(x_0)$, $(df/dx)(x_0)$, or $Df(x_0)$, is defined as $\lim_{h\to 0}[f(x_0 + h) - f(x_0)]/h$ if this limit exists.

Differentiation—i.e., calculating the derivative, seldom requires the use of the basic definition but can instead be accomplished through a knowledge of the three basic derivatives, the use of four rules of operation, and a knowledge of how to manipulate functions.



The graph of a function, drawn in black, and a tangent line to that function, drawn in red. The slope of the tangent line equals the derivative of the function at the marked point

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1.3 Differentiation

Differentiation, in Mathematics is the process of finding the derivative, or rate of change, of a function. In contrast to the abstract nature of the theory behind it, the practical technique of differentiation can be carried out by purely algebraic manipulations, using three basic derivatives, four rules of operation, and a knowledge of how to manipulate functions.

The three basic derivatives (D) are:

(1) for algebraic functions, $D(x^n) = nx^{n-1}$, in which n is any real number; (2) for trigonometric functions, $D(\sin x) = \cos x$ and $D(\cos x) = -\sin x$; and(3) for exponential functions, $D(e^x) = e^x$.

For functions built up of combinations of these classes of functions, the theory provides the following basic rules for differentiating the sum, product, or quotient of any two functions $f(x)$ and $g(x)$ the derivatives of which are known (where a and b are constants): $D(af + bg) = aDf + bDg$ (sums); $D(fg) = fDg + gDf$ (products); and $D(f/g) = (gDf - fDg)/g^2$ (quotients).

The other basic rule, called the chain rule, provides a way to differentiate a composite function. If $f(x)$ and $g(x)$ are two functions, the composite function $f(g(x))$ is calculated for a value of x by first evaluating $g(x)$ and then evaluating the function f at this value of $g(x)$; for instance, if $f(x) = \sin x$ and $g(x) = x^2$, then $f(g(x)) = \sin x^2$, while $g(f(x)) = (\sin x)^2$. The chain rule states that the derivative of a composite function is given by a product, as $D(f(g(x))) = Df(g(x)) \cdot Dg(x)$. In words, the first factor on the right, $Df(g(x))$, indicates that the derivative of $Df(x)$ is first found as usual, and then x, wherever it occurs, is replaced by the function $g(x)$. In the example of $\sin x^2$, the rule gives the result $D(\sin x^2) = D\sin(x^2) \cdot D(x^2) = (\cos x^2) \cdot 2x$.

In the German mathematician Gottfried Wilhelm Leibniz's notation, which uses d/dx in place of D and thus allows differentiation with respect to different variables to be made explicit, the chain rule takes the more memorable "symbolic cancellation" form: $d(f(g(x)))/dx = df/dg \cdot dg/dx$.

II. REVIEW OF RELATED LITERATURE

The concept of a derivative in the sense of a tangent line is a very old one, familiar to ancient Greek mathematicians such as Euclid (c. 300 BC), Archimedes (c. 287–212 BC) and Apollonius of Perga (c. 262–190 BC). Archimedes also made use of indivisibles, although these were primarily used to study areas and volumes rather than derivatives and tangents.

The use of infinitesimals to study rates of change can be found in Indian mathematics, perhaps as early as 500 AD, when the astronomer and mathematician Aryabhata (476–550) used infinitesimals to study the orbit of the Moon. The use of infinitesimals to compute rates of change was developed significantly by Bhāskara II (1114–1185); indeed, it has been arguedthat many of the key notions of differential calculus can be found in his work, such as "Rolle's theorem".

The mathematician, Sharaf al-Dīn al-Tūsī (1135–1213), in his *Treatise on Equations*, established conditions for some cubic equations to have solutions, by finding the maxima of appropriate cubic polynomials. He obtained, for example, that the maximum (for positive x) of the cubic $ax^2 - x^3$ occurs when $x = 2a / 3$, and concluded therefrom that the equation $ax^2 = x^3 + c$ has exactly one positive solution when $c = 4a^3 / 27$, and two positive solutions whenever $0 < c < 4a^3 / 27$. The historian of science, Roshdi Rashed, has argued that al-Tūsī must have used the derivative of the cubic to obtain this result. Rashed's conclusion has been contested by other scholars, however, who argue that he could have obtained the result by other methods which do not require the derivative of the function to be known.

The modern development of calculus is usually credited to Isaac Newton (1643–1727) and Gottfried Wilhelm Leibniz (1646–1716), who provided independent and unified approaches to differentiation and derivatives. The key insight, however, that earned them this credit, was the fundamental theorem of calculus relating differentiation and integration: this rendered obsolete most previous methods for computing areas and volumes which had not been significantly extended since the time of Ibn al-Haytham (Alhazen). For their ideas on derivatives, both Newton and

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Leibniz built on significant earlier work by mathematicians such as Pierre de Fermat (1607-1665), Isaac Barrow (1630–1677), René Descartes (1596–1650), Christiaan Huygens (1629–1695), Blaise Pascal (1623–1662) and John Wallis (1616–1703). Regarding Fermat's influence, Newton once wrote in a letter that "*I had the hint of this method [of fluxions] from Fermat's way of drawing tangents, and by applying it to abstract equations, directly and invertedly, I made it general.*" Isaac Barrow is generally given credit for the early development of the derivative. Nevertheless, Newton and Leibniz remain key figures in the history of differentiation, not least because Newton was the first to apply differentiation to theoretical physics, while Leibniz systematically developed much of the notation still used today.

Since the 17th century many mathematicians have contributed to the theory of differentiation. In the 19th century, calculus was put on a much more rigorous footing by mathematicians such as Augustin Louis Cauchy (1789–1857), Bernhard Riemann (1826–1866), and Karl Weierstrass (1815–1897). It was also during this period that the differentiation was generalized to Euclidean space and the complex plane.

III. L'HÔPITAL'S RULE

L'Hôpital's rule, in Analysis, procedure of differentialcalculus for evaluating indeterminate forms such as 0/0 and ∞/∞ when they result from an attempt to find a limit. It is named for the French mathematician Guillaume-François-Antoine, marquis de L'Hôpital, who purchased the formula from his teacher the Swiss mathematician Johann Bernoulli. L'Hôpital published the formula in *L'Analyse des infiniment petits pour l'intelligence des lignescourbes* (1696), the first textbook on differential calculus.

L'Hôpital's rule states that, when the limit of $f(x)/g(x)$ is indeterminate, under certain conditions it can be obtained by evaluating the limit of the quotient of the derivatives of f and g (i.e., $f'(x)/g'(x)$). If this result is indeterminate, the procedure can be repeated.

IV. QUOTIENT RULE

Quotient rule is the rule for finding the derivative of a quotient of two functions. If both f and g are differentiable, then so is the quotient $f(x)/g(x)$. In abbreviated notation, it says $(f/g)' = (gf' - fg')/g^2$.

In Mathematics, Differential Calculus is a subfield of Calculus that studies the rates at which quantities change. It is one of the two traditional divisions of calculus, the other being Integral Calculus—the study of the area beneath a curve.

The primary objects of study in differential calculus are the derivative of a function, related notions such as the differential, and their applications. The derivative of a function at a chosen input value describes the rate of change of the function near that input value. The process of finding a derivative is called **differentiation**. Geometrically, the derivative at a point is the slope of the tangent line to the graph of the function at that point, provided that the derivative exists and is defined at that point. For a real-valued function of a single real variable, the derivative of a function at a point generally determines the best linear approximation to the function at that point.

Differential calculus and integral calculus are connected by the fundamental theorem of calculus, which states that differentiation is the reverse process to integration.

Differentiation has applications in nearly all quantitative disciplines. In physics, the derivative of the displacement of a moving body with respect to time is the velocity of the body, and the derivative of the velocity with respect to time is acceleration. The derivative of the momentum of a body with respect to time equals the force applied to the body; rearranging this derivative statement leads to the famous $\mathbf{F} = m\mathbf{a}$ equation associated with Newton's second law of motion. The reaction rate of a chemical reaction is a derivative. In operations research, derivatives determine the most efficient ways to transport materials and design factories.

Derivatives are frequently used to find the maxima and minima of a function. Equations involving derivatives are called differential equations and are fundamental in describing natural phenomena. Derivatives and their generalizations appear in many fields of mathematics, such as complex analysis, functional analysis, differential geometry, measure theory, and abstract algebra.

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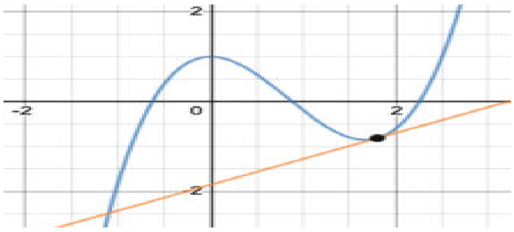
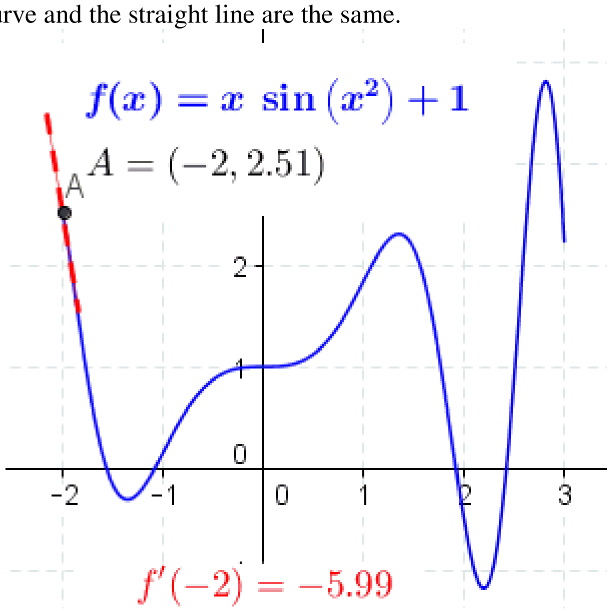
V. DERIVATIVE



In the graph, the slope of the curve and the straight line are the same.



The derivative at different points of a differentiable function

However, many functions cannot be differentiated as easily as polynomial functions, meaning that sometimes further techniques are needed to find the derivative of a function. These techniques include the chain rule, product rule, and quotient rule. Other functions cannot be differentiated at all, giving rise to the concept of differentiability.

A closely related concept to the derivative of a function is its differential. When x and y are real variables, the derivative of f at x is the slope of the tangent line to the graph of f at x. Because the source and target of f are one-dimensional, the derivative of f is a real number. If x and y are vectors, then the best linear approximation to the graph of f depends on how f changes in several directions at once. Taking the best linear approximation in a single direction determines a partial derivative, which is usually denoted $\partial y/\partial x$. The linearization of f in all directions at once is called the total derivative.

VI. APPLICATIONS OF DERIVATIVES

6.1 Optimization

If f is a differentiable function on $\mathbb{R}$ (or an open interval) and x is a local maximum or a local minimum of function f, then the derivative of the function f at x is zero. Points where $f'(x) = 0$ are called *critical points* or *stationary points* (and the value of f at x is called a *critical value*). If f is not assumed to be everywhere differentiable, then points at which it fails to be differentiable are also designated critical points.

If f is twice differentiable, then conversely, a critical point x of function f can be analysed by considering the second derivative of function f at x:

- if it is positive, x is a local minimum;

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- if it is negative, x is a local maximum;
- if it is zero, then x could be a local minimum, a local maximum, or neither. (For example, $f(x) = x^3$ has a critical point at $x = 0$, but it has neither a maximum nor a minimum there, whereas $f(x) = \pm x^4$ has a critical point at $x = 0$ and a minimum and a maximum, respectively, there.)

This is called the second derivative test. An alternative approach, called the first derivative test, involves considering the sign of the f' on each side of the critical point.

Taking derivatives and solving for critical points is therefore often a simple way to find local minima or maxima, which can be useful in optimization. By the extreme value theorem, a continuous function on a closed interval must attain its minimum and maximum values at least once. If the function is differentiable, the minima and maxima can only occur at critical points or endpoints.

This also has applications in graph sketching: once the local minima and maxima of a differentiable function have been found, a rough plot of the graph can be obtained from the observation that it will be either increasing or decreasing between critical points.

In higher dimensions, a critical point of a scalar valued function is a point at which the gradient is zero. The second derivative test can still be used to analyse critical points by considering the eigenvalues of the Hessian matrix of second partial derivatives of the function at the critical point. If all of the eigenvalues are positive, then the point is a local minimum; if all are negative, it is a local maximum. If there are some positive and some negative eigenvalues, then the critical point is called a "saddle point", and if none of these cases hold (i.e., some of the eigenvalues are zero) then the test is considered to be inconclusive.

6.2 Calculus of variations

One example of an optimization problem is: Find the shortest curve between two points on a surface, assuming that the curve must also lie on the surface. If the surface is a plane, then the shortest curve is a line. But if the surface is, for example, egg-shaped, then the shortest path is not immediately clear. These paths are called geodesics, and one of the most fundamental problems in the calculus of variations is finding geodesics. Another example is: Find the smallest area surface filling in a closed curve in space. This surface is called a minimal surface and it, too, can be found using the calculus of variations.

6.3 Physics

Calculus is of vital importance in physics: many physical processes are described by equations involving derivatives, called differential equations. Physics is particularly concerned with the way quantities change and develop over time, and the concept of the '**time derivative**' — the rate of change over time — is essential for the precise definition of several important concepts. In particular, the time derivatives of an object's position are significant in Newtonian physics:

- velocity is the derivative (with respect to time) of an object's displacement (distance from the original position)
- acceleration is the derivative (with respect to time) of an object's velocity, that is, the second derivative (with respect to time) of an object's position.

6.4 Differential equations

A differential equation is a relation between a collection of functions and their derivatives. An ordinary differential equation is a differential equation that relates functions of one variable to their derivatives with respect to that variable. A partial differential equation is a differential equation that relates functions of more than one variable to their partial derivatives. Differential equations arise naturally in the physical sciences, in mathematical modelling, and within mathematics itself. For example, Newton's second law, which describes the relationship between acceleration and force, can be stated as the ordinary differential equation.

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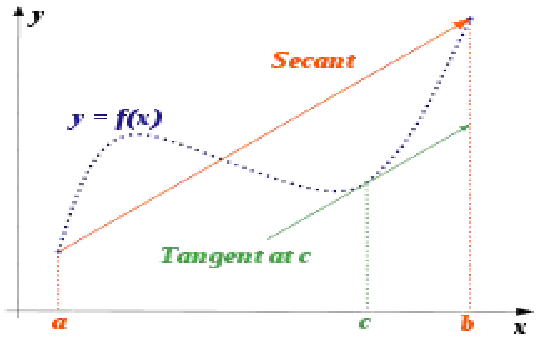
6.5 Mean value theorem



The mean value theorem gives a relationship between values of the derivative and values of the original function. If $f(x)$ is a real-valued function and a and b are numbers with $a<b$, then the mean value theorem says that under mild hypotheses, the slope between the two points $(a, f(a))$ and $(b, f(b))$ is equal to the slope of the tangent line to f at some point c between a and b. In other words,

In practice, what the mean value theorem does is control a function in terms of its derivative. For instance, suppose that f has derivative equal to zero at each point. This means that its tangent line is horizontal at every point, so the function should also be horizontal. The mean value theorem proves that this must be true: The slope between any two points on the graph offunction f must equal the slope of one of the tangent lines of function f. All of those slopes are zero, so any line from one point on the graph to another point will also have slope zero. But that says that the function does not move up or down, so it must be a horizontal line. More complicated conditions on the derivative lead to less precise but still highly useful information about the original function.

6.6 Taylor polynomials and Taylor series

The derivative gives the best possible linear approximation of a function at a given point, but this can be very different from the original function. One way of improving the approximation is to take a quadratic approximation. That is to say, the linearization of a real-valued function $f(x)$ at the point x_0 is a linear polynomial$a + b(x - x_0)$, and it may be possible to get a better approximation by considering a quadratic polynomial $a + b(x - x_0) + c(x - x_0)^2$. Still better might be a cubic polynomial $a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^3$, and this idea can be extended to arbitrarily high degree polynomials. For each one of these polynomials, there should be a best possible choice of coefficients a, b, c, and d that makes the approximation as good as possible.

In the neighbourhood of x_0, for a the best possible choice is always $f(x_0)$, and for b the best possible choice is always $f'(x_0)$. For c, d, and higher-degree coefficients, these coefficients are determined by higher derivatives of f. c should always be $f''(x_0)/2$, and d should always be $f'''(x_0)/3!$. Using these coefficients gives the **Taylor polynomial**of f. The Taylor polynomial of degree d is the polynomial of degree d which best approximates f, and its coefficients can be found by a generalization of the above formulas. Taylor's theorem gives a precise bound on how good the approximation is. If f is a polynomial of degree less than or equal to d, then the Taylor polynomial of degree d equals f.

The limit of the Taylor polynomials is an infinite series called the **Taylor series**. The Taylor series is frequently a very good approximation to the original function. Functions which are equal to their Taylor series are called analytic functions. It is impossible for functions with discontinuities or sharp corners to be analytic; moreover, there exist smooth functions which are also not analytic.

6.7 Implicit function theorem

Some natural geometric shapes, such as circles, cannot be drawn as the graph of a function. For instance, if $f(x, y) = x^2 + y^2 - 1$, then the circle is the set of all pairs (x, y) such that $f(x, y) = 0$. This set is called the zero set of f, and is not the same as the graph of f, which is a paraboloid. The implicit function theorem converts relations such as $f(x, y) = 0$ into functions. It states that if f is continuously differentiable, then around most points, the zero set of f looks like graphs of

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functions pasted together. The points where this is not true are determined by a condition on the derivative of function *f*. The circle, for instance, can be pasted together from the graphs of the two functions $\pm\sqrt{1 - x^2}$. In a neighbourhood of every point on the circle except (−1, 0) and (1, 0), one of these two functions has a graph that looks like the circle. (These two functions also happen to meet (−1, 0) and (1, 0), but this is not guaranteed by the implicit function theorem.)

The implicit function theorem is closely related to the inverse function theorem, which states when a function looks like graphs of invertible functions pasted together.

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